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Hypersymplectic Lie algebras

Adrián Andrada [1]

CIEM, FaMAF, Universidad Nacional de Córdoba, Ciudad Universitaria, (5000) Córdoba, Argentina

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Abstract

We characterize real Lie algebras carrying a hypersymplectic structure as bicrossproducts of two symplectic Lie algebras endowed with a compatible flat torsion-free connection. In particular, we obtain the classification of all hypersymplectic structures on 4-dimensional Lie algebras, and we describe the associated metrics on the corresponding Lie groups.

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1. Introduction

A hypersymplectic structure on a manifold is a complex product structure, i.e. a pair $\{J, E\}$ of a complex structure and a product structure that anticommute, together with a compatible metric such that the associated 2-forms are closed. This notion is similar to that of a hyperkähler structure, where the base manifold carries a hypercomplex structure, i.e. a pair $\{J_1, J_2\}$ of anticommuting complex structures.

Hypersymplectic structures were introduced by Hitchin in [11], and they are also referred to as *neutral hyperkähler* structures in [12] and as *parahyperkähler* structures in [1]. Hypersymplectic structures on manifolds have become an important subject of study lately, due mainly to its applications in theoretical physics (especially in dimension 4). See for instance [5], where there is a discussion on the relationship between hypersymplectic metrics and the $N = 2$ superstring.

E-mail address: andrada@mate.uncor.edu.

[1] Present address: The Abdus Salam International Centre for Theoretical Physics, Strada Costiera 11, 34014, Trieste, Italy. Tel.: +39 040 2240220; fax: +39 040 224163.

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Hypersymplectic metrics on a manifold are Ricci-flat and the associated holonomy group is contained in the real symplectic group.

In [12], the compact complex surfaces which admit hypersymplectic structures are determined. These complex surfaces are either complex tori or primary Kodaira surfaces; it is also shown when the hypersymplectic metrics on these surfaces are flat. In [8], examples of (non-flat) hypersymplectic structures are given on Kodaira manifolds, which are special compact quotients of 2-step nilpotent Lie groups. These hypersymplectic structures are not invariant by the nilpotent Lie group. In [3], examples of hypersymplectic nilmanifolds are exhibited, where in this case the hypersymplectic structures are invariant by the action of the nilpotent Lie group and, moreover, the complex structure is abelian. The corresponding hypersymplectic metrics are not necessarily flat. In [1], a classification is given of the symmetric spaces admitting a hypersymplectic structure and it is shown that these spaces are Osserman.

The main goal of this paper is to give the classification, up to equivalence, of all left invariant hypersymplectic structures on 4-dimensional Lie groups. These Lie groups will provide examples of hypersymplectic structures in non-compact manifolds, since their underlying differentiable manifolds are diffeomorphic to $\mathbb{R}^4$. In order to perform this classification, we begin in Section 3 the study of hypersymplectic structures on real Lie algebras. We obtain that, associated to a hypersymplectic structure $\{J, E, g\}$ on a Lie algebra $\mathfrak{g}$, there are two triples $(\mathfrak{g}_+, \nabla^+, \omega_+)$, $(\mathfrak{g}_-, \nabla^-, \omega_-)$, where $\mathfrak{g}_\pm$ are Lie subalgebras of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ and $\mathfrak{g}_- = J\mathfrak{g}_+$, $\nabla^\pm$ is a flat torsion-free connection on $\mathfrak{g}_\pm$ and $\omega_\pm$ is a symplectic form on $\mathfrak{g}_\pm$ such that $\omega_+(x, y) = \omega_-(Jx, Jy)$ for all $x, y \in \mathfrak{g}_+$ and $\nabla^\pm\omega_\pm = 0$. Conversely, we show that, in certain cases, given two triples $(\mathfrak{g}_+, \nabla^+, \omega_+)$ and $(\mathfrak{g}_-, \nabla^-, \omega_-)$ satisfying the same conditions as above, we can obtain a hypersymplectic structure on $\mathfrak{g}_+ \oplus \mathfrak{g}_-$ (direct sum of vector spaces). This result will be used in the 4-dimensional case. We also deal with equivalences between hypersymplectic structures.

Next, in Section 4, we give the first steps in order to achieve the classification mentioned above, namely, we determine the flat torsion-free connections on the 2-dimensional Lie algebras which are compatible with a symplectic form and obtain their equivalence classes.

In Section 5, we show that a complex product structure on a Lie algebra admits at most one compatible metric (up to a multiplicative constant) and we state our main result which says that, aside from the abelian Lie algebra, there are only three Lie algebras which admit a hypersymplectic structure. One of them is a central extension of the 3-dimensional Heisenberg algebra $\mathfrak{h}_3$; the second one is an extension of $\mathbb{R}^3$ and the third one is an extension of $\mathfrak{h}_3$. We also parameterize the underlying complex product structures with the corresponding hypersymplectic metrics, pointing out when these metrics are flat and/or complete. The proof of this theorem is the content of Section 6. As an illustration we exhibit the following (non-isometric) hypersymplectic metrics on $\mathbb{R}^4$ with canonical global coordinates t, x, y, z:

(i) $g = \mathrm{d}t^2 + \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$ (flat and complete).
(ii) $g = \mathrm{e}^{-t}\mathrm{d}t(\mathrm{d}z - \frac{1}{2}x\mathrm{d}y + \frac{1}{2}y\mathrm{d}x) + \mathrm{e}^{-t}\mathrm{d}x\mathrm{d}y$ (flat but not complete).
(iii) $g = \mathrm{e}^{t}\mathrm{d}t\mathrm{d}z + \mathrm{e}^{2t}\mathrm{d}z^2 - \mathrm{d}x\mathrm{d}y + \mathrm{e}^{2t}\mathrm{d}y^2$ (neither flat nor complete).

2. Preliminaries

We start recalling some definitions which will be used throughout this work. All Lie algebras will be finite dimensional and defined over $\mathbb{R}$.

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and suppose that G admits a left invariant (affine) connection ∇, i.e., each left translation $L_g : G \to G, x \mapsto gx$ is an affine transformation of G.

In this case, if $X, Y \in \mathfrak{g}$ are two left invariant vector fields on G then $\nabla_X Y \in \mathfrak{g}$ is also left invariant. Moreover, there is a one–one correspondence between the set of left invariant connections on G and the set of bilinear functions $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (see [10, p. 102]). Accordingly, in this work we will consider the following notion of connection on a Lie algebra:

Definition 1. A connection on a Lie algebra $\mathfrak{g}$ is a bilinear form $\nabla : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. The connection is called *torsion-free* if $\nabla_x y - \nabla_y x = [x, y]$ for all $x, y \in \mathfrak{g}$ and is called *flat* if the curvature R of ∇ is identically zero, where $R(x, y) = \nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x,y]}$, $x, y \in \mathfrak{g}$.

We recall that flat torsion-free connections on a Lie algebra are also known as "left-symmetric algebra" (LSA) structures. It is known that the completeness of the left invariant connection ∇ on G can be studied by considering simply the corresponding connection on the Lie algebra $\mathfrak{g}$. Indeed, the left invariant connection ∇ on G will be (geodesically) complete if and only if the differential equation on $\mathfrak{g}$

$$\dot{x}(t) = -\nabla_{x(t)} x(t) \tag{1}$$

admits solutions $x(t) \in \mathfrak{g}$ defined for all $t \in \mathbb{R}$ (see for instance [7] or [9]).

We also recall the definition of complex structures and product structures on a Lie algebra, which are modelled on the corresponding notions for smooth manifolds.

An *almost complex structure* on a Lie algebra $\mathfrak{g}$ is a linear endomorphism $J : \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying $J^2 = -\mathbf{1}$. If J satisfies the condition

$$J[x, y] = [Jx, y] + [x, Jy] + J[Jx, Jy] \qquad \text{for all } x, y \in \mathfrak{g}, \tag{2}$$

we will say that J is *integrable* and we will call it a *complex structure* on $\mathfrak{g}$. Note that the dimension of a Lie algebra carrying an almost complex structure must be even.

Next, an *almost product structure* on $\mathfrak{g}$ is a linear endomorphism $E : \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying $E^2 = \mathbf{1}$ (and not equal to $\pm\mathbf{1}$). It is said to be *integrable* if

$$E[x, y] = [Ex, y] + [x, Ey] - E[Ex, Ey] \qquad \text{for all } x, y \in \mathfrak{g}. \tag{3}$$

An integrable almost product structure will be called a *product structure*. If $\mathfrak{g}_\pm$ is the eigenspace of $\mathfrak{g}$ associated to the eigenvalue ± 1 of E, then the integrability of E is equivalent to the fact of $\mathfrak{g}_\pm$ being Lie subalgebras of $\mathfrak{g}$. If $\dim \mathfrak{g}_+ = \dim \mathfrak{g}_-$, the product structure E is called a *paracomplex structure* [14,15]. In this case, $\mathfrak{g}$ has even dimension.

An appropriate combination of these two structures on Lie algebras is called a complex product structure, and its definition is given below.

Definition 2 (*[4]*). A *complex product structure* on the Lie algebra $\mathfrak{g}$ is a pair $\{J, E\}$ of a complex structure J and a product structure E satisfying $JE = -EJ$.

Complex product structures on Lie algebras have been studied in [4], from where we recall some of their main properties. The condition $JE = -EJ$ implies that J is an isomorphism (as vector spaces) between $\mathfrak{g}_+$ and $\mathfrak{g}_-$, the eigenspaces corresponding to the eigenvalues $+1$ and -1 of E, respectively; thus, E is in fact a paracomplex structure on $\mathfrak{g}$. Every complex product structure on $\mathfrak{g}$ has therefore an associated *double Lie algebra* $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$, i.e., $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are Lie subalgebras of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ (direct sum of vector spaces) and $\mathfrak{g}_- = J\mathfrak{g}_+$, where $E|_{\mathfrak{g}_+} = \mathbf{1}$, $E|_{\mathfrak{g}_-} = -\mathbf{1}$.

The complex product structure $\{J, E\}$ on $\mathfrak{g}$ determines uniquely a torsion-free connection ∇^{CP} on $\mathfrak{g}$ such that $\nabla^{\mathrm{CP}} J = \nabla^{\mathrm{CP}} E = 0$, where these equations mean that

$$\nabla^{\mathrm{CP}}_x Jy = J\nabla^{\mathrm{CP}}_x y, \qquad \nabla^{\mathrm{CP}}_x Ey = E\nabla^{\mathrm{CP}}_x y$$

for all $x, y \in \mathfrak{g}$. As a consequence, we note that $\nabla^{\mathrm{CP}}_x y \in \mathfrak{g}_\pm$ for any $x \in \mathfrak{g}$ and $y \in \mathfrak{g}_\pm$. Take now $x \in \mathfrak{g}_+$, $y \in \mathfrak{g}_-$. Since ∇^{CP} has no torsion, we obtain that

$$[x, y] = -\nabla^{\mathrm{CP}}_y x + \nabla^{\mathrm{CP}}_x y \in \mathfrak{g}_+ \oplus \mathfrak{g}_- \tag{4}$$

is the decomposition of $[x, y]$ into components, according to the splitting $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$. The connection ∇^{CP} restricts to *flat* torsion-free connections on $\mathfrak{g}_+$ and $\mathfrak{g}_-$, say ∇^+ and ∇^-, respectively. Thus, we have that ∇^{CP} is flat if and only if $R(x_+, x_-) = 0$ for all $x_+ \in \mathfrak{g}_+$, $x_- \in \mathfrak{g}_-$, where R is the curvature of ∇^{CP}.

3. Hypersymplectic structures on Lie algebras

We study in this section a special kind of metrics on a Lie algebra with a complex product structure, just as hyperhermitian and hyperkähler metrics appear in the context of hypercomplex structures.

Let $\mathfrak{g}$ be a Lie algebra endowed with a complex product structure $\{J, E\}$ and let g be a metric on $\mathfrak{g}$, i.e., g is a non-degenerate symmetric bilinear form $g : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$. We will say that g is *compatible* with the complex product structure if, for all $x, y \in \mathfrak{g}$,

$$g(Jx, Jy) = g(x, y), \qquad g(Ex, Ey) = -g(x, y). \tag{5}$$

Let $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ denote the double Lie algebra associated to the complex product structure $\{J, E\}$, where $\mathfrak{g}_- = J\mathfrak{g}_+$ and let g be a compatible metric. Then it follows easily that the subalgebras $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are isotropic subspaces of $\mathfrak{g}$ with respect to g, i.e., $g(\mathfrak{g}_+, \mathfrak{g}_+) = 0$, $g(\mathfrak{g}_-, \mathfrak{g}_-) = 0$. From this it is clear that $\mathfrak{g}_+^\perp = \mathfrak{g}_+$ and $\mathfrak{g}_-^\perp = \mathfrak{g}_-$ and also that the signature of g is (m, m), where $\dim \mathfrak{g} = 2m$.

Let us now define the following bilinear forms on $\mathfrak{g}$:

$$\omega_1(x, y) = g(Jx, y), \qquad \omega_2(x, y) = g(Ex, y), \qquad \omega_3(x, y) = g(JEx, y) \tag{6}$$

for $x, y \in \mathfrak{g}$. Using (5), it is readily verified that these forms are in fact skew-symmetric, so that $\omega_i \in \bigwedge^2 \mathfrak{g}^*$ for $i = 1, 2, 3$. Note that these forms are non-degenerate, since g is non-degenerate and J and E are isomorphisms. In the following result, whose proof is straightforward, we show the existing relationships between these 2-forms on $\mathfrak{g}$ and the decomposition of this Lie algebra induced by the product structure E.

Lemma 3. *The 2-forms ω_i, $i = 1, 2, 3$, on $\mathfrak{g}$ satisfy the following properties:*

(i) $\omega_1(x, y) = \omega_1(Jx, Jy) = \omega_1(Ex, Ey)$ *for all* $x, y \in \mathfrak{g}$, *whence* $\omega_1(x, y) = 0$ *for* $x \in \mathfrak{g}_+$, $y \in \mathfrak{g}_-$.

(ii) $-\omega_2(x, y) = \omega_2(Jx, Jy) = \omega_2(Ex, Ey)$ *for all* $x, y \in \mathfrak{g}$, *whence* $\omega_2(x, y) = 0$ *for* $x, y \in \mathfrak{g}_+$ *or* $x, y \in \mathfrak{g}_-$.

(iii) $\omega_3(x, y) = -\omega_3(Jx, Jy) = \omega_3(Ex, Ey)$ *for all* $x, y \in \mathfrak{g}$, *whence* $\omega_3(x, y) = 0$ *for* $x \in \mathfrak{g}_+$, $y \in \mathfrak{g}_-$.

Let ω_+ and ω_- denote the restriction of ω_1 to $\mathfrak{g}_+$ and $\mathfrak{g}_-$, respectively. From (i) of the previous lemma and the fact that ω_1 is non-degenerate, it is easy to see that both ω_+ and ω_- are non-degenerate. Hence, $m = \dim \mathfrak{g}_+ = \dim \mathfrak{g}_-$ must be an even number, say $m = 2n$, and therefore $\dim \mathfrak{g} = 4n$ and the signature of g is $(2n, 2n)$.

From Lemma 3(i) we obtain that

$$\omega_+(x, y) = \omega_-(Jx, Jy) \tag{7}$$

for all $x, y \in \mathfrak{g}_+$. Actually, the 2-forms ω_1, ω_2 and ω_3 can be written in terms exclusively of ω_+, since we have the identities

$$\omega_1(x + Jx', y + Jy') = \omega_+(x, y) + \omega_+(x', y'), \tag{8}$$

$$\omega_2(x + Jx', y + Jy') = -\omega_+(x, y') + \omega_+(y, x'), \tag{9}$$

$$\omega_3(x + Jx', y + Jy') = \omega_+(x, y) - \omega_+(x', y') \tag{10}$$

for all $x, y, x', y' \in \mathfrak{g}_+$. Eqs. (8)–(10) follow easily from (6), Lemma 3 and the relations $\omega_2(u, Jv) = -\omega_1(u, v)$ and $\omega_3(u, v) = \omega_1(u, v)$ for all $u, v \in \mathfrak{g}_+$.

Let us recall now that given a 2-form ω on a Lie algebra $\mathfrak{g}$, there is an associated 3-form $\mathrm{d}\omega \in \bigwedge^3 \mathfrak{g}^*$ given by

$$(\mathrm{d}\omega)(x, y, z) = -\omega([x, y], z) + \omega([x, z], y) - \omega([y, z], x)$$

for all $x, y, z \in \mathfrak{g}$. The 2-form ω is called *closed* if $\mathrm{d}\omega = 0$; if ω is non-degenerate and closed, it is called a *symplectic form* on $\mathfrak{g}$.

Naturally, we are mainly interested in the case when all of the 2-forms given in (6) are closed and hence symplectic. We introduce therefore the following definition, equivalent to the one given by Hitchin in [11].

Definition 4. Let $\{J, E\}$ be a complex product structure on the Lie algebra $\mathfrak{g}$ and let g be a metric on $\mathfrak{g}$ compatible with $\{J, E\}$. If the 2-forms $\omega_i \in \bigwedge^2 \mathfrak{g}^*$ defined in (6) are closed, we will say that $\{J, E, g\}$ is a *hypersymplectic structure* on $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ will be referred to as a *hypersymplectic Lie algebra* and g will be called a *hypersymplectic metric*.

The surprising fact is that if one of the 2-forms ω_1 or ω_3 is closed, then all three of these 2-forms are closed, as the following result shows.

Proposition 5. *Let $\{J, E\}$ be a complex product structure on $\mathfrak{g}$ with associated double Lie algebra $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$. Let ∇^+ and ∇^- denote the flat torsion-free connections on $\mathfrak{g}_+$ and $\mathfrak{g}_-$ induced by ∇^{CP}. Suppose g is a compatible metric on $\mathfrak{g}$ and let ω_i, $i = 1, 2, 3$, be the 2-forms on $\mathfrak{g}$ given by (6) and ω_+ and ω_- be as above. Then the following statements are equivalent:*

(i) *ω_1 is closed;*
(ii) *ω_3 is closed;*
(iii) *$\nabla^+\omega_+ = 0$ and $\nabla^-\omega_- = 0$.*

Furthermore, if one of the conditions above holds, then

(iv) *ω_2 is closed.*

Remark 6. We recall that a 2-form ω on a Lie algebra $\mathfrak{h}$ is parallel with respect to a connection ∇ on $\mathfrak{h}$, i.e. $\nabla\omega = 0$, if the condition $\omega(\nabla_x y, z) = \omega(\nabla_x z, y)$ holds for all $x, y, z \in \mathfrak{h}$. We will also say that ∇ and ω are compatible.

Proof. (i) $\Leftrightarrow$ (iii). Let us suppose first that (i) holds. For $x, y \in \mathfrak{g}_+$ and $z = Ju$ with $u \in \mathfrak{g}_+$ we have that

$$
\begin{aligned}
(\mathrm{d}\omega_1)(x, y, Ju) &= \omega_1([x, Ju], y) - \omega_1([y, Ju], x) \\
&= -\omega_1(\nabla^{\mathrm{CP}}_{Ju} x, y) + \omega_1(\nabla^{\mathrm{CP}}_{Ju} y, x) \\
&= -\omega_1(\nabla^{\mathrm{CP}}_{Ju} Jx, Jy) + \omega_1(\nabla^{\mathrm{CP}}_{Ju} Jy, Jx) \\
&= -\omega_-(\nabla^-_{Ju} Jx, Jy) + \omega_-(\nabla^-_{Ju} Jy, Jx),
\end{aligned} \tag{11}
$$

using (4). As $\mathrm{d}\omega_1 = 0$, we obtain that $\nabla^-\omega_- = 0$. If we consider now $x \in \mathfrak{g}_+$ and $y = Jv, z = Ju$ with $u, v \in \mathfrak{g}_+$, we have that

$$
\begin{aligned}
(\mathrm{d}\omega_1)(x, Jv, Ju) &= -\omega_1([x, Jv], Ju) + \omega_1([x, Ju], Jv) \\
&= -\omega_1(\nabla^{\mathrm{CP}}_{x} Jv, Ju) + \omega_1(\nabla^{\mathrm{CP}}_{x} Ju, Jv) \\
&= -\omega_1(\nabla^{\mathrm{CP}}_{x} v, u) + \omega_1(\nabla^{\mathrm{CP}}_{x} u, v) \\
&= -\omega_+(\nabla^+_{x} v, u) + \omega_+(\nabla^+_{x} u, v),
\end{aligned} \tag{12}
$$

using again (4). As $\mathrm{d}\omega_1 = 0$, we obtain that $\nabla^+\omega_+ = 0$. Thus, (iii) holds.

Conversely, let us suppose that (iii) holds. We note first that, as ∇^+ and ∇^- are torsion-free, one obtains that $\mathrm{d}\omega_+ = 0$ and $\mathrm{d}\omega_- = 0$. Suppose first that $x, y, z \in \mathfrak{g}_+$. Then $\mathrm{d}\omega_1(x, y, z) = \mathrm{d}\omega_+(x, y, z) = 0$ since ω_+ is closed. Similarly, for $x, y, z \in \mathfrak{g}_-$, we have $\mathrm{d}\omega_1(x, y, z) = \mathrm{d}\omega_-(x, y, z) = 0$ since ω_- is closed. Next, if $x, y \in \mathfrak{g}_+$ and $z = Ju$ with $u \in \mathfrak{g}_+$, from Eq. (11) and $\nabla^-\omega_- = 0$, we have that $(\mathrm{d}\omega_1)(x, y, Ju) = 0$. Finally, if $x \in \mathfrak{g}_+$ and $y = Jv, z = Ju$ with $u, v \in \mathfrak{g}_+$, from Eq. (12) and $\nabla^+\omega_+ = 0$, we have that $(\mathrm{d}\omega_1)(x, Jv, Ju) = 0$. Therefore, $\mathrm{d}\omega_1 = 0$.

(ii) $\Leftrightarrow$ (iii). The proof is similar to the proof of (i) $\Leftrightarrow$ (iii).

(iii) $\Rightarrow$ (iv). If $x, y, z \in \mathfrak{g}_+$ or $x, y, z \in \mathfrak{g}_-$, then $(\mathrm{d}\omega_2)(x, y, z) = 0$, because of Lemma 3(ii). If $x, y \in \mathfrak{g}_+$ and $z = Ju$ with $u \in \mathfrak{g}_+$, then we have

$$
\begin{aligned}
(\mathrm{d}\omega_2)(x, y, Ju) &= -\omega_2([x, y], Ju) + \omega_2([x, Ju], y) - \omega_2([y, Ju], x) \\
&= -\omega_2([x, y], Ju) + \omega_2(\nabla^{\mathrm{CP}}_{x} Ju, y) - \omega_2(\nabla^{\mathrm{CP}}_{y} Ju, x) \\
&= \omega_1([x, y], u) - \omega_1(\nabla^{\mathrm{CP}}_{x} u, y) + \omega_1(\nabla^{\mathrm{CP}}_{y} u, x) \\
&= \omega_+([x, y], u) - \omega_+(\nabla^+_{x} u, y) + \omega_+(\nabla^+_{y} u, x) \\
&= \omega_+([x, y], u) - \omega_+(\nabla^+_{x} y, u) + \omega_+(\nabla^+_{y} x, u) \\
&= 0
\end{aligned}
$$

because of (4), (iii) and since ∇^+ is torsion-free. Now suppose that $x \in \mathfrak{g}_+$ and $y = Jv, z = Ju$ with $u, v \in \mathfrak{g}_+$. We have that

$$
\begin{aligned}
(\mathrm{d}\omega_2)(x, Jv, Ju) &= -\omega_2([x, Jv], Ju) + \omega_2([x, Ju], Jv) - \omega_2([Jv, Ju], x) \\
&= \omega_2(\nabla^{\mathrm{CP}}_{Jv} x, Ju) - \omega_2(\nabla^{\mathrm{CP}}_{Ju} x, Jv) + \omega_2(J[Jv, Ju], Jx) \\
&= -\omega_1(\nabla^{\mathrm{CP}}_{Jv} x, u) + \omega_1(\nabla^{\mathrm{CP}}_{Ju} x, v) - \omega_1(J[Jv, Ju], x) \\
&= -\omega_1(\nabla^{\mathrm{CP}}_{Jv} Jx, Ju) + \omega_1(\nabla^{\mathrm{CP}}_{Ju} Jx, Jv) + \omega_1([Jv, Ju], Jx) \\
&= -\omega_1(\nabla^{\mathrm{CP}}_{Jv} Ju, Jx) + \omega_1(\nabla^{\mathrm{CP}}_{Ju} Jv, Jx) + \omega_1([Jv, Ju], Jx) \\
&= -\omega_-(\nabla^-_{Jv} Ju, Jx) + \omega_-(\nabla^-_{Ju} Jv, Jx) + \omega_1([Jv, Ju], Jx) \\
&= 0
\end{aligned}
$$

because of (4), (iii) and since ∇^- is torsion-free. Hence, (iv) holds and the proof of the proposition is complete. □

At this point we can state a characterization of hypersymplectic Lie algebras in terms of two Lie algebras equipped with a flat torsion-free connection and a parallel symplectic form.

Theorem 7. *Let $\{J, E, g\}$ be a hypersymplectic structure on $\mathfrak{g}$ with $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ the double Lie algebra associated to the complex product structure $\{J, E\}$. Then we have two triples $(\mathfrak{g}_+, \nabla^+, \omega_+)$ and $(\mathfrak{g}_-, \nabla^-, \omega_-)$ where $\nabla^\pm$ is a flat torsion-free connection and $\omega_\pm$ is a parallel symplectic form on $\mathfrak{g}_\pm$. These symplectic forms are related by: $\omega_+(x, y) = \omega_-(Jx, Jy)$ for $x, y \in \mathfrak{g}_+$.*

Conversely, suppose $(\mathfrak{g}_+, \nabla^+, \omega_+)$ and $(\mathfrak{g}_-, \nabla^-, \omega_-)$ are two triples consisting of a Lie algebra, a flat torsion-free connection and a parallel symplectic form. If there exists a linear isomorphism $\varphi : \mathfrak{g}_+ \to \mathfrak{g}_-$ such that

(i) *the representations $\rho : \mathfrak{g}_+ \to \mathfrak{gl}(\mathfrak{g}_-)$ and $\mu : \mathfrak{g}_- \to \mathfrak{gl}(\mathfrak{g}_+)$ defined by*

$$\rho(x)a = \varphi\nabla_x\varphi^{-1}(a), \quad \mu(a)x = \varphi^{-1}\nabla'_a\varphi(x)$$

satisfy

$$\rho(x)[a, b] - [\rho(x)a, b] - [a, \rho(x)b] + \rho(\mu(a)x)b - \rho(\mu(b)x)a = 0,$$
$$\mu(a)[x, y] - [\mu(a)x, y] - [x, \mu(a)y] + \mu(\rho(x)a)y - \mu(\rho(y)a)x = 0,$$

for all $x, y \in \mathfrak{g}_+$ and $a, b \in \mathfrak{g}_-$;

(ii) *$\omega(x, y) = \omega'(\varphi(x), \varphi(y))$ for all $x, y \in \mathfrak{g}_+$;*

then the vector space $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ admits a Lie bracket extending the Lie brackets on $\mathfrak{g}_+$ and $\mathfrak{g}_-$ and there is a hypersymplectic structure on $\mathfrak{g}$ such that its associated double Lie algebra is $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$.

Remark 8. Note that $\{\mathfrak{g}, E, \omega_2\}$ is a parakähler Lie algebra (see [13]).

Proof. The first part of the theorem follows from Proposition 5 and the discussion previous to it.

Now we prove the converse. Condition (i) means that $(\mathfrak{g}_+, \mathfrak{g}_-, \rho, \mu)$ is a *matched pair of Lie algebras* (see [16]). Thus, the bracket on $\mathfrak{g}$ given by

$$[(x, a), (y, b)] = ([x, y] + \mu(a)y - \mu(b)x, [a, b] + \rho(x)b - \rho(y)a),$$

for $x, y \in \mathfrak{g}_+$ and $a, b \in \mathfrak{g}_-$ satisfies the Jacobi identity; $\mathfrak{g}$ with this Lie algebra structure will be denoted $\mathfrak{g} = \mathfrak{g}_+ \bowtie_\mu^\rho \mathfrak{g}_-$ (or simply $\mathfrak{g} = \mathfrak{g}_+ \bowtie \mathfrak{g}_-$) and will be called the *bicrossproduct* of $\mathfrak{g}_+$ and $\mathfrak{g}_-$. Observe that $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are Lie subalgebras of $\mathfrak{g}$. Taking into account the definition of ρ and μ, we get that

$$[(x, 0), (0, a)] = (-\varphi^{-1}\nabla'_a\varphi(x), \varphi\nabla_x\varphi^{-1}(a))$$

for $x \in \mathfrak{g}_+$ and $a \in \mathfrak{g}_-$. It has already been proved in [4] that $\mathfrak{g} = \mathfrak{g}_+ \bowtie \mathfrak{g}_-$ admits a complex product structure $\{J, E\}$, where the endomorphisms J and E are defined by

$$J(x, a) = (-\varphi^{-1}(a), \varphi(x)), \qquad E|_{\mathfrak{g}_+} = \mathbf{1}, \quad E|_{\mathfrak{g}_-} = -\mathbf{1},$$

for $x \in \mathfrak{g}_+, a \in \mathfrak{g}_-$. Furthermore, if ∇^{CP} denotes the torsion-free connection associated to $\{J, E\}$, then the restrictions of ∇^{CP} to $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are precisely the original connections ∇ and ∇', respectively.

We proceed now to define a metric g on $\mathfrak{g}$ which will be shown to be hypersymplectic. Let g be given by

$$g(\mathfrak{g}_+, \mathfrak{g}_+) = 0, \qquad g(\mathfrak{g}_-, \mathfrak{g}_-) = 0, \qquad g((x, a), (y, b)) = \omega(\varphi^{-1}(b), x) + \omega(\varphi^{-1}(a), y)$$

for $x, y \in \mathfrak{g}_+$, $a, b \in \mathfrak{g}_-$. It is clear that g is a metric on $\mathfrak{g}$. We should check now that it satisfies (5). We begin with

$$\begin{aligned} g(J(x, a), J(y, b)) &= g((-\varphi^{-1}(a), \varphi(x)), (-\varphi^{-1}(b), \varphi(y))) \\ &= -\omega(y, \varphi^{-1}(a)) - \omega(x, \varphi^{-1}(b)) \\ &= \omega(\varphi^{-1}(a), y) + \omega(\varphi^{-1}(b), x) \\ &= g((x, a), (y, b)) \end{aligned}$$

and now

$$\begin{aligned} g(E(x, a), E(y, b)) &= g((x, -a), (y, -b)) \\ &= -\omega(\varphi^{-1}(b), x) - \omega(\varphi^{-1}(a), y) \\ &= -g((x, a), (y, b)). \end{aligned}$$

Thus (5) holds and g is compatible with $\{J, E\}$.

To see that with this metric we obtain a hypersymplectic structure on $\mathfrak{g}$, we only have to see that (iii) of Proposition 5 holds. Let us determine firstly the 2-form ω_1 on $\mathfrak{g}$:

$$\begin{aligned} \omega_1((x, a), (y, b)) &= g(J(x, a), (y, b)) \\ &= g((-\varphi^{-1}(a), \varphi(x)), (y, b)) \\ &= \omega(\varphi^{-1}(b), -\varphi^{-1}(a)) + \omega(x, y) \\ &= \omega(x, y) + \omega'(a, b). \end{aligned}$$

Therefore, the restrictions of ω_1 to $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are precisely the original symplectic forms ω and ω', respectively. As $\nabla^+\omega_+ = \nabla\omega = 0$ and $\nabla^-\omega_- = \nabla'\omega' = 0$, we have that g is a hypersymplectic metric on $\mathfrak{g}$. □

Any metric g on a Lie algebra $\mathfrak{g}$ determines by left translations a left invariant metric on G, where G is the only simply connected Lie group with $\mathrm{L}(G) = \mathfrak{g}$. It is easy to verify that the Levi-Civita connection on the manifold G is also left invariant, and hence it is determined by its values at $\mathfrak{g} \cong \mathrm{T}_e G$. Therefore, the metric g on $\mathfrak{g}$ determines a connection ∇^g on $\mathfrak{g}$, also called the Levi-Civita connection associated to g. This Levi-Civita connection is the only connection on $\mathfrak{g}$ such that (i) it is torsion-free, and (ii) the endomorphisms ∇^g_x, $x \in \mathfrak{g}$, are skew-adjoint with respect to g. Just as in the positive definite case, in the neutral setting one can prove the following equivalences:

Proposition 9. *Let $\mathfrak{g}$ be a Lie algebra with a complex product structure $\{J, E\}$ and a compatible metric g. Let ∇^g denote the Levi-Civita connection on $\mathfrak{g}$ associated to g and let ω_i, $i = 1, 2, 3$, be the 2-forms on $\mathfrak{g}$ given in (6). Then the following statements are equivalent:*

(i) *The metric g is hypersymplectic, i.e.,* $\mathrm{d}\omega_i = 0$ *for $i = 1, 2, 3$.*
(ii) *The endomorphisms J and E are ∇^g-parallel: $\nabla^g J = \nabla^g E = 0$.*
(iii) *The 2-forms ω_i, $i = 1, 2, 3$, are ∇^g-parallel: $\nabla^g \omega_i = 0$ for $i = 1, 2, 3$.*

Corollary 10. *If $\{J, E, g\}$ is a hypersymplectic structure on the Lie algebra $\mathfrak{g}$, then $\nabla^g = \nabla^{\mathrm{CP}}$, where ∇^g is the Levi-Civita connection associated to g and ∇^{CP} is the complex product connection associated to $\{J, E\}$.*

Proof. Recalling that ∇^{CP} is the only torsion-free connection with respect to which J and E are parallel, and taking into account the equivalence (i) $\Leftrightarrow$ (ii) of Proposition 9, we obtain that $\nabla^g = \nabla^{\mathrm{CP}}$. □

We consider now the question of equivalences between hypersymplectic structures. We have the following definition.

Definition 11. Let $\{J, E, g\}$ and $\{J', E', g'\}$ be hypersymplectic structures on the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ respectively. These structures are said to be *equivalent* if there exists a Lie algebra isomorphism $\xi : \mathfrak{g} \longrightarrow \mathfrak{g}'$ such that

$$\xi J = J'\xi, \qquad \xi E = E'\xi \qquad \text{and} \qquad g'(\xi x, \xi y) = g(x, y) \tag{13}$$

for all $x, y \in \mathfrak{g}$.

Remark 12. The first two conditions in (13) mean that the underlying complex product structures $\{J, E\}$ and $\{J', E'\}$ are equivalent. The third condition means that ξ is an isometry between g and g'.

Lemma 13. *With notation as in the previous definition, let ∇^g and $\nabla^{g'}$ be the Levi-Civita connection of g and g' respectively. Then ξ gives an equivalence between these two connections. Furthermore, if ω_i, $i = 1, 2, 3$, are given as in (6) and ω'_i, $i = 1, 2, 3$, are defined similarly for $\mathfrak{g}'$, then $\omega_i(x, y) = \omega'_i(\xi x, \xi y)$ for all $x, y \in \mathfrak{g}$.*

Proof. According to Corollary 10, the Levi-Civita connection ∇^g of g is the only torsion-free connection on $\mathfrak{g}$ such that $\nabla^g J = \nabla^g E = 0$, and a similar statement holds for $\nabla^{g'}$. We would like to show that $\xi \nabla^g_x y = \nabla^{g'}_{\xi x} \xi y$ for all $x, y \in \mathfrak{g}$. To see this, define a connection $\tilde{\nabla}$ on $\mathfrak{g}$ by

$$\tilde{\nabla}_x y := \xi^{-1} \nabla^{g'}_{\xi x} \xi y, \qquad x, y \in \mathfrak{g}.$$

Let us see that it is torsion-free:

$$\tilde{\nabla}_x y - \tilde{\nabla}_y x = \xi^{-1}(\nabla^{g'}_{\xi x}\xi y - \nabla^{g'}_{\xi y}\xi x) = \xi^{-1}[\xi x, \xi y] = [x, y]$$

for any $x, y \in \mathfrak{g}$. Let us verify now that J is $\tilde{\nabla}$-parallel.

$$\tilde{\nabla}_x J y = \xi^{-1}\nabla^{g'}_{\xi x}\xi J y = \xi^{-1}\nabla^{g'}_{\xi x} J'\xi y = \xi^{-1} J' \nabla^{g'}_{\xi x}\xi y = J\xi^{-1}\nabla^{g'}_{\xi x}\xi y = J\tilde{\nabla}_x y$$

for all $x, y \in \mathfrak{g}$. In the same way, it can be seen that $\tilde{\nabla} E = 0$. By uniqueness, we have that $\nabla^g = \tilde{\nabla}$ and hence ∇^g and $\nabla^{g'}$ are equivalent.

Let us check now the assertions about the symplectic forms. Let us consider first the 2-form ω_1. We have

$$\omega_1(x, y) = g(Jx, y) = g'(\xi J x, \xi y) = g'(J'\xi x, \xi y) = \omega'_1(\xi x, \xi y)$$

for all $x, y \in \mathfrak{g}$. In a similar fashion one can prove the corresponding statements for ω_2 and ω_3. □

Motivated by the previous result, we introduce the following definition.

Definition 14. Let $\mathfrak{g}$ be a Lie algebra equipped with a connection ∇ and a symplectic form ω such that $\nabla\omega = 0$, and similarly for a Lie algebra $\mathfrak{g}'$ with ∇' and ω'. We will say that $(\mathfrak{g}, \nabla, \omega)$ and $(\mathfrak{g}', \nabla', \omega')$ are *symplectically equivalent* if there exists a Lie algebra isomorphism $\xi : \mathfrak{g} \to \mathfrak{g}'$ such that

$$\xi \nabla_x y = \nabla'_{\xi x} \xi y, \qquad \omega(x, y) = \omega'(\xi x, \xi y) \qquad \text{for all } x, y \in \mathfrak{g}.$$

Proposition 15. *Keep the notation from Theorem 7. Suppose that $(\mathfrak{g}_+, \nabla, \omega)$ is symplectically equivalent to $(\mathfrak{g}_+, \overline{\nabla}, \overline{\omega})$, where $\overline{\nabla}$ is a flat torsion-free connection on $\mathfrak{g}_+$ and $\overline{\omega}$ is a symplectic form on $\mathfrak{g}_+$ such that $\overline{\nabla}\overline{\omega} = 0$. Similarly, let $(\mathfrak{g}_-, \nabla', \omega')$ be symplectically equivalent to $(\mathfrak{g}_-, \overline{\nabla}', \overline{\omega}')$. Then we obtain a matched pair of Lie algebras $(\mathfrak{g}_+, \mathfrak{g}_-, \overline{\rho}, \overline{\mu})$ and the bicrossproduct $\overline{\mathfrak{g}} = \mathfrak{g}_+ \bowtie^{\overline{\rho}}_{\overline{\mu}} \mathfrak{g}_-$ has a hypersymplectic structure equivalent to the one on $\mathfrak{g} = \mathfrak{g}_+ \bowtie^{\rho}_{\mu} \mathfrak{g}_-$.*

Remark 16. See the proof of Theorem 7 for the definition of a matched pair of Lie algebras.

Proof. Let $\xi : \mathfrak{g}_+ \to \mathfrak{g}_+$ and $\xi' : \mathfrak{g}_- \to \mathfrak{g}_-$ be the Lie algebra isomorphisms which give the symplectic equivalences between $(\mathfrak{g}_+, \nabla, \omega)$ and $(\mathfrak{g}_+, \overline{\nabla}, \overline{\omega})$ and between $(\mathfrak{g}_-, \nabla', \omega')$ and $(\mathfrak{g}_-, \overline{\nabla}', \overline{\omega}')$, respectively. Consider now the linear isomorphism $\psi : \mathfrak{g}_+ \to \mathfrak{g}_-$ given by $\psi = \xi'\varphi\xi^{-1}$. Associated to the isomorphism ψ we have the representations $\overline{\rho} : \mathfrak{g}_+ \to \mathfrak{gl}(\mathfrak{g}_-)$ and $\overline{\mu} : \mathfrak{g}_- \to \mathfrak{gl}(\mathfrak{g}_+)$ defined by

$$\overline{\rho}(x)a = \psi \overline{\nabla}_x \psi^{-1}(a), \qquad \overline{\mu}(a)x = \psi^{-1} \overline{\nabla}'_a \psi(x).$$

It is easily verified that $(\mathfrak{g}_+, \mathfrak{g}_-, \overline{\rho}, \overline{\mu})$ is a matched pair of Lie algebras, using that $(\mathfrak{g}_+, \mathfrak{g}_-, \rho, \mu)$ is another matched pair of Lie algebras. We may form now the bicrossproduct Lie algebras $\mathfrak{g} = \mathfrak{g}_+ \bowtie^{\rho}_{\mu} \mathfrak{g}_-$ and $\overline{\mathfrak{g}} = \mathfrak{g}_+ \bowtie^{\overline{\rho}}_{\overline{\mu}} \mathfrak{g}_-$. Furthermore, it is easy to see that $\overline{\omega}(x, y) = \overline{\omega}'(\psi(x), \psi(y))$ for all $x, y \in \mathfrak{g}_+$. From Theorem 7, both $\mathfrak{g}$ and $\overline{\mathfrak{g}}$ have a hypersymplectic structure. Consider now the linear isomorphism $\eta := \xi \oplus \xi' : \mathfrak{g} \longrightarrow \overline{\mathfrak{g}}$; it is straightforward to verify that η defines an equivalence between the hypersymplectic structures on $\mathfrak{g}$ and $\overline{\mathfrak{g}}$. □

3.1. At the Lie group level

Let $\{J, E, g\}$ be a hypersymplectic structure on the Lie algebra $\mathfrak{g}$, and let G be the simply connected Lie group with Lie algebra $\mathfrak{g}$, where we consider an element of $\mathfrak{g}$ as a left invariant vector field on G. The hypersymplectic structure on $\mathfrak{g}$ determines, by left translations, a hypersymplectic structure on G, still denoted by $\{J, E, g\}$. This means that J is a complex structure, E is a product structure and g is a pseudo-Riemannian metric on G, compatible with J and E, such that $\nabla^g J = \nabla^g E = 0$. Equivalently, the 2-forms $\omega_i \in \Omega^2(G), i = 1, 2, 3$, defined as in (6), are symplectic (parallel) forms. The metric g is automatically Ricci flat and its holonomy is contained in $Sp(n, \mathbb{R})$, where $\dim G = 4n$. If $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is the associated double Lie algebra, with $\mathfrak{g}_- = J\mathfrak{g}_+$, let G_+ and G_- denote the connected Lie subgroups of G with Lie algebras $\mathfrak{g}_+$ and $\mathfrak{g}_-$, respectively. The decomposition $\mathfrak{g} = \mathfrak{g}_+ \bowtie \mathfrak{g}_-$ determines naturally two complementary distributions on G, both of them involutive, and the leaves of the foliations $\mathcal{F}^+$ and $\mathcal{F}^-$ determined by these distributions are totally real submanifolds of the complex manifold (G, J). Moreover, these leaves are totally geodesic and flat with respect to the canonical torsion-free connection ∇^{CP} determined by $\{J, E\}$ (which coincides with the Levi-Civita connection of the hypersymplectic metric g). The foliations $\mathcal{F}^{\pm}$ are symplectic with respect to any of the symplectic forms ω_1 and ω_3, whereas they are lagrangian with respect to the remaining form ω_2. It is easy to see that the leaf of $\mathcal{F}^{\pm}$ passing through $x \in G$ is $xG_{\pm}$, and it is well known that this leaf is embedded if and only if $G_{\pm}$ is closed in G. If the Lie group G admits a lattice Γ, then the hypersymplectic structure on G induces one on the compact manifold $\Gamma \setminus G$.

4. Symplectic flat torsion-free connections on $\mathbb{R}^2$ and $\mathfrak{aff}(\mathbb{R})$

In the next sections, we will determine all the 4-dimensional Lie algebras which carry a hypersymplectic structure. In order to do so, we will need to know all the flat torsion-free connections that preserve a symplectic form on the 2-dimensional Lie algebras. We recall that, up to isomorphism, there are only two 2-dimensional Lie algebras, namely, $\mathbb{R}^2$ and the Lie algebra $\mathfrak{aff}(\mathbb{R})$, which has a basis $\{e_1, e_2\}$ such that $[e_1, e_2] = e_2$. $\mathfrak{aff}(\mathbb{R})$ is the Lie algebra of the Lie group $Aff(\mathbb{R})$ of affine motions of the real line.

We start with the abelian Lie algebra $\mathbb{R}^2$.

Theorem 17. *Let $\mathbb{R}^2 = \operatorname{span}\{e_1, e_2\}$ denote the 2-dimensional abelian Lie algebra and let $\omega = e^1 \wedge e^2$ be the canonical symplectic form on $\mathbb{R}^2$. Then the only non-zero flat torsion-free connections ∇ on $\mathbb{R}^2$ such that $\nabla\omega = 0$ are the following:*

(a) *For $\alpha \neq 0$:*

$$\nabla_{e_1} = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}, \qquad \nabla_{e_2} = 0;$$

(b) *For $\alpha \neq 0$:*

$$\nabla_{e_1} = 0, \qquad \nabla_{e_2} = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix};$$

(c) *For $\alpha \neq 0, \beta \neq 0$:*

$$\nabla_{e_1} = \begin{pmatrix} \alpha & -\dfrac{\alpha^2}{\beta} \\ \beta & -\alpha \end{pmatrix}, \qquad \nabla_{e_2} = \begin{pmatrix} -\dfrac{\alpha^2}{\beta} & \dfrac{\alpha^3}{\beta^2} \\ -\alpha & \dfrac{\alpha^2}{\beta} \end{pmatrix}.$$

Proof. Let us denote

$$\begin{aligned} \nabla_{e_1} e_1 &= a e_1 + b e_2, \\ \nabla_{e_1} e_2 &= c e_1 + d e_2 = \nabla_{e_2} e_1, \\ \nabla_{e_2} e_2 &= g e_1 + h e_2, \end{aligned}$$

with $a, b, c, d, g, h \in \mathbb{R}$. Since ∇ is flat, we have that $\nabla_{e_1}\nabla_{e_2} = \nabla_{e_2}\nabla_{e_1}$, and from this condition we obtain that

$$\begin{aligned} & bg = cd, \\ & bc - bh + d^2 - ad = 0, \\ & ag - dg + ch - c^2 = 0. \end{aligned} \tag{14}$$

Now, the condition $\nabla\omega = 0$ holds if and only if $\omega(\nabla_x y, z) = \omega(\nabla_x z, y)$ for all $x, y, z \in \mathbb{R}^2$. From this we get

$$d = -a \qquad \text{and} \qquad h = -c.$$

Substituting into (14), we obtain

$$\begin{aligned} bg &= -ac, \\ a^2 &= -bc, \\ c^2 &= ag. \end{aligned} \tag{15}$$

If $a = 0$, then $c = 0$ and $bg = 0$. As $\nabla \neq 0$, then $b \neq 0$ or $g \neq 0$. If $b \neq 0$, then $g = 0$ and ∇ is of type (a) in the statement. If $g \neq 0$, then $b = 0$ and ∇ is of type (b) in the statement.

Let us suppose now $a \neq 0$. Then $bcg \neq 0$ and from (15) we obtain $c = -\frac{a^2}{b}$ and $g = \frac{a^3}{b^2}$. Therefore, ∇ is of type (c) in the statement. $\square$

In the next proposition we study the equivalences among the connections obtained in Theorem 17.

Proposition 18. *Let ∇ be a non-zero flat torsion-free connection on $\mathbb{R}^2$ and ω a ∇-parallel symplectic form on $\mathbb{R}^2$. Then $(\mathbb{R}^2, \nabla, \omega)$ is symplectically equivalent to $(\mathbb{R}^2, \nabla^0, e^1 \wedge e^2)$, where $\{e_1, e_2\}$ is a suitable basis of $\mathbb{R}^2$, $\{e^1, e^2\}$ is the dual basis and ∇^0 is given by*

$$\nabla^0_{e_1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \nabla^0_{e_2} = 0;$$

This flat torsion-free connection on $\mathbb{R}^2$ is complete.

Proof. There exists a basis $\{e_1, e_2\}$ of $\mathbb{R}^2$ such that $\omega = e^1 \wedge e^2$. Since $\nabla\omega = 0$, the connection ∇ must be one of those given by Theorem 17.

Let us suppose first that ∇ is of type (a) in Theorem 17. The linear isomorphism of $\mathbb{R}^2$ which gives the symplectic equivalence between ∇ and ∇^0 is given by

$$\xi = \begin{pmatrix} \alpha^{1/3} & 0 \\ 0 & \alpha^{-1/3} \end{pmatrix}$$

in the ordered basis $\{e_1, e_2\}$.

Suppose now that the connection ∇ is of type (b) in Theorem 17. The linear isomorphism of $\mathbb{R}^2$ which gives the symplectic equivalence between ∇ and ∇^0 is given by

$$\xi = \begin{pmatrix} 0 & -\alpha^{1/3} \\ \alpha^{-1/3} & 0 \end{pmatrix}$$

in the ordered basis $\{e_1, e_2\}$.

Finally, if ∇ is of type (c) in Theorem 17, we may take the following isomorphism of $\mathbb{R}^2$:

$$\xi = \begin{pmatrix} \beta^{1/3} & -\alpha\beta^{-2/3} \\ 0 & \beta^{-1/3} \end{pmatrix}.$$

The verification of all these statements is simple. We would like now to check that the connection ∇^0 is complete. In order to do so, we will use (1). Let $x(t) = a_1(t)e_1 + a_2(t)e_2$ be a curve on $\mathfrak{g}$ which satisfies $\dot{x}(t) = -\nabla^0_{x(t)} x(t)$. Thus, we obtain the system of differential equations

$$\begin{cases} \dot{a_1} = 0, \\ \dot{a_2} = -a_1^2. \end{cases}$$

The solutions of this system are clearly defined for every $t \in \mathbb{R}$ and therefore ∇^0 is complete. $\square$

Remark 19. In [18], a classification of flat torsion-free connections (up to equivalence) on the abelian Lie algebra $\mathbb{R}^2$ is given. The flat torsion-free connection ∇^0 from Proposition 18 belongs to the class A_4 of that classification.

Next, we move on to consider the other 2-dimensional Lie algebra, $\mathfrak{aff}(\mathbb{R})$.

Theorem 20. *Let* $\mathfrak{aff}(\mathbb{R}) = \operatorname{span}\{e_1, e_2\}$ *denote the 2-dimensional Lie algebra with Lie bracket* $[e_1, e_2] = e_2$ *and let* $\omega = e^1 \wedge e^2$ *be the canonical symplectic form on* $\mathfrak{aff}(\mathbb{R})$. *Then the only flat torsion-free connections* ∇ *on* $\mathfrak{aff}(\mathbb{R})$ *such that* $\nabla\omega = 0$ *are the following:*

(a) *For* $\alpha \in \mathbb{R}$*:*

$$\nabla_{e_1} = \begin{pmatrix} -1 & 0 \\ \alpha & 1 \end{pmatrix}, \qquad \nabla_{e_2} = 0;$$

(b) *For* $\alpha \in \mathbb{R}$*:*

$$\nabla_{e_1} = \begin{pmatrix} -\frac{1}{2} & 0 \\ \alpha & \frac{1}{2} \end{pmatrix}, \qquad \nabla_{e_2} = \begin{pmatrix} 0 & 0 \\ -\frac{1}{2} & 0 \end{pmatrix}.$$

Proof. Let us denote

$$\begin{aligned} \nabla_{e_1} e_1 &= ae_1 + be_2, \\ \nabla_{e_1} e_2 &= ce_1 + de_2, \\ \nabla_{e_2} e_2 &= ge_1 + he_2, \end{aligned}$$

with $a, b, c, d, g, h \in \mathbb{R}$. Since ∇ is torsion-free, we have

$$\nabla_{e_2} e_1 = ce_1 + (d-1)e_2.$$

The condition $\nabla\omega = 0$ implies that $d = -a$ and $h = -c$. Taking this into account and using that ∇ is flat, we obtain the following equations

$$\begin{aligned} & c(a+2) + bg = 0, \\ & g(2a-1) - 2c^2 = 0, \\ & 2bc + (a+1)(2a+1) = 0. \end{aligned} \tag{16}$$

From the third equation in (16) we get

$$2a^2 + 3a + (2bc+1) = 0. \tag{17}$$

Also, from (16) we see immediately that $a \neq \frac{1}{2}$. Hence $g = \frac{2c^2}{2a-1}$ and substituting into the first equation we have

$$c\left((a+2) + \frac{2bc}{2a-1}\right) = 0.$$

If $c \neq 0$, then $(a+2)(2a-1) + 2bc = 0$ and hence $2a^2 + 3a + 2bc - 2 = 0$, which combined with (17) yields a contradiction. Thus, $c = 0$ and the system (16) becomes

$$\begin{aligned} & bg = 0, \\ & g(2a-1) = 0, \\ & (a+1)(2a+1) = 0. \end{aligned} \tag{18}$$

Therefore, $g = 0$ (since $a \neq \frac{1}{2}$), $b \in \mathbb{R}$ is arbitrary and $a = -1$ or $a = -\frac{1}{2}$. In the first case, we obtain a connection of type (a) and in the second case we obtain a connection of type (b). The proof is complete. □

In the next proposition we deal with the equivalences of the connections obtained in Theorem 20.

Proposition 21. *Let ∇ be a flat torsion-free connection on $\mathfrak{aff}(\mathbb{R})$ and ω a ∇-parallel symplectic form on $\mathfrak{aff}(\mathbb{R})$. Then $(\mathfrak{aff}(\mathbb{R}), \nabla, \omega)$ is symplectically equivalent to either $(\mathfrak{aff}(\mathbb{R}), \nabla^1, e^1 \wedge e^2)$ or $(\mathfrak{aff}(\mathbb{R}), \nabla^2, e^1 \wedge e^2)$, where $\{e_1, e_2\}$ is a suitable basis of $\mathfrak{aff}(\mathbb{R})$, $\{e^1, e^2\}$ is the dual basis and ∇^1, ∇^2 are given by:*

$$\nabla^1_{e_1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \nabla^1_{e_2} = 0;$$

and

$$\nabla^2_{e_1} = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \qquad \nabla^2_{e_2} = \begin{pmatrix} 0 & 0 \\ -\frac{1}{2} & 0 \end{pmatrix}.$$

None of the connections ∇^1 and ∇^2 on $\mathfrak{aff}(\mathbb{R})$ is complete.

Proof. Let $\{\tilde{e}_1, \tilde{e}_2\}$ be a basis of $\mathfrak{aff}(\mathbb{R})$ such that $[\tilde{e}_1, \tilde{e}_2] = \tilde{e}_2$. There exists $\lambda \neq 0$ such that $\omega = \lambda(\tilde{e}^1 \wedge \tilde{e}^2)$. Set $e_1 := \tilde{e}_1$, $e_2 := \lambda\tilde{e}_2$. We have then $[e_1, e_2] = e_2$ and

$$\omega = \lambda(\tilde{e}^1 \wedge \tilde{e}^2) = \lambda(e^1 \wedge \lambda^{-1} e^2) = e^1 \wedge e^2.$$

So, we have $\nabla(e^1 \wedge e^2) = 0$, and then ∇ must be one of the flat torsion-free connections given in Theorem 20.

Let us suppose first that ∇ is of type (a) in Theorem 20. The linear isomorphism of $\mathfrak{aff}(\mathbb{R})$ which gives the symplectic equivalence between ∇ and ∇^1 is given by

$$\xi = \begin{pmatrix} 1 & 0 \\ \frac{1}{2}\alpha & 1 \end{pmatrix}$$

in the ordered basis $\{e_1, e_2\}$.

If we take now a connection ∇ of type (b) in Theorem 20, the linear isomorphism of $\mathfrak{aff}(\mathbb{R})$ giving the symplectic equivalence between ∇ and ∇^2 is

$$\xi = \begin{pmatrix} 1 & 0 \\ 2\alpha & 1 \end{pmatrix}$$

in the ordered basis $\{e_1, e_2\}$.

Next, we observe that ∇^1 and ∇^2 are not equivalent. If they were, the subspaces $W_1 = \{x \in \mathfrak{aff}(\mathbb{R}) : \nabla^1_x \equiv 0\}$ and $W_2 = \{x \in \mathfrak{aff}(\mathbb{R}) : \nabla^2_x \equiv 0\}$ of $\mathfrak{aff}(\mathbb{R})$ should be isomorphic. However, it is clear that $\dim W_1 = 1$ while $W_2 = \{0\}$. Thus, these two connections are not equivalent.

Finally, we show that these connections are not complete. Suppose $x(t) = a_1(t)e_1 + a_2(t)e_2$ is a curve on $\mathfrak{aff}(\mathbb{R})$ that satisfies $\dot{x}(t) = -\nabla^1_{x(t)} x(t)$. Thus, we obtain the system of differential equations

$$\begin{cases} \dot{a_1} = a_1^2, \\ \dot{a_2} = -a_1 a_2. \end{cases}$$

From the first equation in the system we obtain that $a_1(t)$ cannot be defined in the whole real line; thus ∇^1 is not complete. Analogously, if $x(t) = a_1(t)e_1 + a_2(t)e_2$ is a curve on $\mathfrak{aff}(\mathbb{R})$ that satisfies $\dot{x}(t) = -\nabla^2_{x(t)}x(t)$, we have the system

$$\begin{cases} \dot{a_1} = \frac{1}{2}a_1^2, \\ \dot{a_2} = 0. \end{cases}$$

We obtain again that $a_1(t)$ cannot be defined in the whole real line; thus ∇^2 is not complete. □

5. Hypersymplectic 4-dimensional Lie algebras

In this section we will determine all 4-dimensional hypersymplectic Lie algebras, by employing Theorem 7. We will also be able to obtain a parameterization of the hypersymplectic structures, up to equivalence.

We fix first some notation on 4-dimensional Lie algebras and Lie groups which will be needed in what follows.

(i) Let $\mathfrak{g}_0^h$ be defined by $\mathfrak{g}_0^h = \operatorname{span}\{v_0, v_1, v_2, v_3\}$ with $[v_1, v_2] = v_3$. This Lie algebra is a central extension of the 3-dimensional Heisenberg algebra and it is the only 2-step nilpotent 4-dimensional Lie algebra.

Let G_0^h denote the simply connected Lie group corresponding to $\mathfrak{g}_0^h$. It is well known that G_0^h is diffeomorphic to $\mathbb{R}^4$ via the exponential map and, by standard computations, we can find global coordinates t, x, y, z on G_0^h such that the left invariant 1-forms $\{v^0, v^1, v^2, v^3\}$ dual to the left invariant vector fields $\{v_i\}_{0\le i\le 3}$ are given by

$$v^0 = \mathrm{d}t, \qquad v^1 = \mathrm{d}x, \qquad v^2 = \mathrm{d}y, \qquad v^3 = -x\mathrm{d}y + \mathrm{d}z.$$

(ii) Let $\mathfrak{g}_1^h$ be defined by $\mathfrak{g}_1^h = \operatorname{span}\{v_0, v_1, v_2, v_3\}$ with $[v_0, v_1] = v_1, [v_0, v_2] = -v_2$ and $[v_0, v_3] = -v_3$. This Lie algebra is an extension of $\mathbb{R}^3$ and it lies in the class $\mathfrak{r}_{4,-1,-1}$ of the classification of 4-dimensional solvable Lie algebras given in [2]. It is 2-step solvable and not unimodular.

Let G_1^h denote the simply connected Lie group corresponding to $\mathfrak{g}_1^h$. It is well known that G_1^h is diffeomorphic to $\mathbb{R}^4$ and, by standard computations, we can find global coordinates t, x, y, z on G_1^h such that the left invariant 1-forms $\{v^0, v^1, v^2, v^3\}$ dual to the left invariant vector fields $\{v_i\}_{0\le i\le 3}$ are given by

$$v^0 = \mathrm{d}t, \qquad v^1 = \mathrm{e}^{-t}\mathrm{d}x, \qquad v^2 = \mathrm{e}^{t}\mathrm{d}y, \qquad v^3 = \mathrm{e}^{t}\mathrm{d}z.$$

(iii) Let $\mathfrak{g}_2^h$ be defined by $\mathfrak{g}_2^h = \operatorname{span}\{v_0, v_1, v_2, v_3\}$ with $[v_0, v_1] = 2v_1, [v_0, v_2] = -v_2, [v_0, v_3] = v_3$ and $[v_1, v_2] = v_3$. This Lie algebra is an extension of $\mathfrak{h}_3$ and it lies in the class $\mathfrak{d}_{4,2}$ of the classification of 4-dimensional solvable Lie algebras given in [2]. It is 3-step solvable and not unimodular.

Let G_2^h denote the simply connected Lie group corresponding to $\mathfrak{g}_2^h$. It is well known that G_2^h is diffeomorphic to $\mathbb{R}^4$ and, by standard computations, we can find global coordinates t, x, y, z on G_2^h such that the left-invariants 1-forms $\{v^0, v^1, v^2, v^3\}$ dual to the left invariant vector fields $\{v_i\}_{0\le i\le 3}$ are given by

$$v^0 = \mathrm{d}t, \qquad v^1 = \mathrm{e}^{-2t}\mathrm{d}x, \qquad v^2 = \mathrm{e}^{t}\mathrm{d}y, \qquad v^3 = \mathrm{e}^{-t}\left(\mathrm{d}z - \frac{1}{2}x\mathrm{d}y + \frac{1}{2}y\mathrm{d}x\right).$$

We shall show next that given a complex product structure on a 4-dimensional Lie algebra, there is only one compatible metric, up to a non-zero constant. A proof of this lemma can also be found in [6].

Lemma 22. *Let $\{J, E\}$ be a complex product structure on a 4-dimensional Lie algebra. If g and h are two metrics on $\mathfrak{g}$ compatible with $\{J, E\}$, then there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $h = \lambda g$.*

Proof. Let $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$, with $\mathfrak{g}_- = J\mathfrak{g}_+$, be the double Lie algebra associated to $\{J, E\}$. Take a basis $\{e_1, e_2\}$ of $\mathfrak{g}_+$ and the corresponding basis $\{f_1 := Je_1, f_2 := Je_2\}$ of $\mathfrak{g}_-$. Since the metric g is compatible with $\{J, E\}$, we know that $g(e_i, e_j) = 0$, $g(f_i, f_j) = 0$ and $g(e_i, f_i) = 0$, for $1 \leq i, j \leq 2$. Also, $g(e_1, f_2) = -g(e_2, f_1) = \alpha$ for some $\alpha \neq 0$, since g is non-degenerate. Hence, $g = \alpha\langle\cdot,\cdot\rangle$, where $\langle\cdot,\cdot\rangle$ is the metric on $\mathfrak{g}$ compatible with $\{J, E\}$ whose only non-zero values are $\langle e_1, f_2\rangle = 1$, $\langle e_2, f_1\rangle = -1$. Similarly, $h = \beta\langle\cdot,\cdot\rangle$ for some $\beta \neq 0$, and the result follows. □

We will also need the following (2×2) matrices, for $\theta \in \mathbb{R}$:

$$\mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \mathbf{A}_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix},$$

$$\mathbf{B}_\theta = \begin{pmatrix} -\sin\theta & 1+\cos\theta \\ 1+\cos\theta & \sin\theta \end{pmatrix}, \qquad \mathbf{C}_\theta = \begin{pmatrix} \sin\theta(1+\cos\theta) & -\cos\theta(1+\cos\theta) \\ \cos\theta(1+\cos\theta) & \sin\theta(1+\cos\theta) \end{pmatrix}.$$

Now we can state the theorem of classification of 4-dimensional hypersymplectic Lie algebras.

Theorem 23. *Let $\mathfrak{g}$ be a 4-dimensional Lie algebra carrying a hypersymplectic structure. Then $\mathfrak{g}$ is isomorphic to either $\mathbb{R}^4$, $\mathfrak{g}_0^h$, $\mathfrak{g}_1^h$ or $\mathfrak{g}_2^h$. Furthermore, the parameterization of the hypersymplectic structures in each case is given by:*

(i) *If $\mathfrak{g} \cong \mathbb{R}^4$, the underlying complex product structure is equivalent to $\{J, E\}$, where*

$$J = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \qquad E = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$$

(with $\mathbf{1}$ the (2×2)-identity matrix) in some ordered basis of $\mathbb{R}^4$. The left invariant hypersymplectic metric on the abelian Lie group $\mathbb{R}^4$ is

$$g = \mathrm{d}t^2 + \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2,$$

which is flat and complete.

(ii) *If $\mathfrak{g} \cong \mathfrak{g}_0^h$, then the underlying complex product structure on $\mathfrak{g}$ is equivalent to one and only one of $\{J^{(0)}, E_\theta^{(0)}\}$, where $J^{(0)}v_1 = v_2$, $J^{(0)}v_3 = v_0$ and*

$$E_\theta^{(0)} = \begin{pmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_\theta \end{pmatrix}$$

for $\theta \in [0, 2\pi)$, in the ordered basis $\{v_1, v_2, v_3, v_0\}$. The corresponding left invariant hypersymplectic metrics on G_0^h are

$$\begin{aligned} g_\theta^{(0)} &= -\cos_{\theta/2} x\mathrm{d}y^2 + \cos_{\theta/2} x\mathrm{d}y\mathrm{d}z + \sin_{\theta/2} \mathrm{d}t\mathrm{d}y \\ &\quad - \sin_{\theta/2} x\mathrm{d}x\mathrm{d}y + \sin_{\theta/2} \mathrm{d}x\mathrm{d}z - \cos_{\theta/2} \mathrm{d}t\mathrm{d}x. \end{aligned}$$

The metrics are flat and complete for all θ, and hence isometric to the canonical neutral metric on $\mathbb{R}^4$ given in (i).

(iii) *If* $\mathfrak{g} \cong \mathfrak{g}_1^h$, *then the underlying complex product structure on* $\mathfrak{g}$ *is equivalent to one and only one of* $\{J^{(1)}, E^{(1)}_{\theta,d}\}$ *or* $\{J^{(1)}, E^{(1)}_1\}$, *where* $J^{(1)}v_0 = v_1$, $J^{(1)}v_2 = v_3$ *and*

$$E^{(1)}_{\theta,d} = \begin{pmatrix} \mathbf{A}_\theta & \mathbf{0} \\ d\mathbf{B}_\theta & \mathbf{E} \end{pmatrix}, \qquad E^{(1)}_1 = \begin{pmatrix} -\mathbf{E} & \mathbf{0} \\ -2\mathbf{E} & \mathbf{E} \end{pmatrix}$$

for $\theta \in [0, 2\pi)$ *and* $d = 0$ *or* $d = 1$, *in the ordered basis* $\{v_0, v_1, v_2, v_3\}$. *The corresponding left invariant hypersymplectic metrics on* G_1^h *are*

$$\begin{aligned} g^{(1)}_{\theta,0} &= \cos_{\theta/2} \mathrm{e}^t \mathrm{d}t\mathrm{d}z + \sin_{\theta/2} \mathrm{d}x\mathrm{d}z + \sin_{\theta/2} \mathrm{e}^t \mathrm{d}t\mathrm{d}y - \cos_{\theta/2} \mathrm{d}x\mathrm{d}y, \\ g^{(1)}_{\theta,1} &= \cos_{\theta/2} \mathrm{e}^t \mathrm{d}t\mathrm{d}z + \sin_{\theta/2} \mathrm{d}x\mathrm{d}z + \cos_{\theta/2} \mathrm{e}^{2t} \mathrm{d}z^2 \\ &\quad + \sin_{\theta/2} \mathrm{e}^t \mathrm{d}t\mathrm{d}y - \cos_{\theta/2} \mathrm{d}x\mathrm{d}y + \cos_{\theta/2} \mathrm{e}^{2t} \mathrm{d}y^2. \end{aligned}$$

and

$$g^{(1)}_1 = \mathrm{d}x\mathrm{d}z + \mathrm{e}^{2t}\mathrm{d}z^2 + \mathrm{e}^t\mathrm{d}t\mathrm{d}y + \mathrm{e}^{2t}\mathrm{d}y^2.$$

These metrics are all non-complete; $g^{(1)}_{\theta,0}$ *and* $g^{(1)}_1$ *are flat, while* $g^{(1)}_{\theta,1}$ *is flat if and only if* $\theta = \pi$.

(iv) *If* $\mathfrak{g} \cong \mathfrak{g}_2^h$, *then the underlying complex product structure on* $\mathfrak{g}$ *is equivalent to one and only one of* $\{J^{(2)}, E^{(2)}_{\theta,d}\}$ *or* $\{J^{(2)}, E^{(2)}_1\}$, *where* $J^{(2)}v_2 = v_0$, $J^{(2)}v_1 = v_3$ *and*

$$E^{(2)}_{\theta,d} = \begin{pmatrix} \mathbf{A}_\theta & \mathbf{0} \\ d\mathbf{C}_\theta & \mathbf{A}_{(-\theta)} \end{pmatrix}, \qquad E^{(2)}_1 = \begin{pmatrix} -\mathbf{E} & \mathbf{0} \\ -2\mathbf{J} & -\mathbf{E} \end{pmatrix}$$

for $\theta \in [0, 2\pi)$ *and* $d = 0$ *or* $d = 1$, *in the ordered basis* $\{v_0, v_2, v_1, v_3\}$. *The corresponding left invariant hypersymplectic metrics on* G_2^h *are*

$$\begin{aligned} g^{(2)}_{\theta,0} &= \mathrm{e}^{-t}\mathrm{d}t\left(\mathrm{d}z - \frac{1}{2}x\mathrm{d}y + \frac{1}{2}y\mathrm{d}x\right) + \mathrm{e}^{-t}\mathrm{d}x\mathrm{d}y, \\ g^{(2)}_{\theta,1} &= \cos_{\theta/2} \mathrm{e}^t \mathrm{d}t\mathrm{d}z + \sin_{\theta/2} \mathrm{d}x\mathrm{d}z + \cos_{\theta/2} \mathrm{e}^{2t} \mathrm{d}z^2 \\ &\quad + \sin_{\theta/2} \mathrm{e}^t \mathrm{d}t\mathrm{d}y - \cos_{\theta/2} \mathrm{d}x\mathrm{d}y + \cos_{\theta/2} \mathrm{e}^{2t} \mathrm{d}y^2, \end{aligned}$$

and

$$\begin{aligned} g^{(2)}_1 &= \mathrm{e}^{-t}\mathrm{d}x\mathrm{d}y + \mathrm{e}^{-4t}\mathrm{d}x^2 + \mathrm{e}^{-t}\mathrm{d}t\left(\mathrm{d}z - \frac{1}{2}x\mathrm{d}y + \frac{1}{2}y\mathrm{d}x\right) \\ &\quad + \mathrm{e}^{-2t}\left(\mathrm{d}z - \frac{1}{2}x\mathrm{d}y + \frac{1}{2}y\mathrm{d}x\right)^2. \end{aligned}$$

These metrics are all non-complete; $g^{(2)}_{\theta,0}$ *and* $g^{(2)}_1$ *are flat, while* $g^{(2)}_{\theta,1}$ *is flat if and only if* $\theta = \pi$.

We end this section with some remarks; the proof of this theorem will be postponed to Section 6.

Remark 24. Note that $E^{(1)}_{\pi,0} = E^{(1)}_{\pi,1}$ and $E^{(2)}_{\pi,0} = E^{(2)}_{\pi,1}$.

Remark 25. (i) Complex structures on 4-dimensional solvable Lie algebras were classified in [17] and [19]. The Lie algebra $\mathfrak{g}_0^h$ lies in the class $S1$ of [19], the algebra $\mathfrak{g}_1^h$ lies in the class $A2\,(\lambda = -1)$ of [17] and finally, $\mathfrak{g}_2^h$ is in the class $H5\,(\lambda_1 = 2, \lambda_2 = -1)$ of [17]. The first two Lie algebras carry only one complex structure, up to equivalence, and they coincide with the complex structures $J^{(0)}$ and $J^{(1)}$ in Theorem 23, respectively. In contrast, $\mathfrak{g}_2^h$ carries two non-equivalent complex structures: one of them coincides with the complex structure $J^{(2)}$ from Theorem 23, while the other one cannot be part of any hypersymplectic structure on $\mathfrak{g}_2^h$.

(ii) A classification of 4-dimensional Lie algebras admitting a complex product structure was given by Blazić and Vukmirović in [6], where they refer to complex product structures as para-hypercomplex structures. The hypersymplectic Lie algebras obtained earlier can be found within this classification.

(iii) The Lie group G_0^h is isomorphic to $H_3 \times \mathbb{R}$, where H_3 is the 3-dimensional Heisenberg group. It is well known that this Lie group admits discrete subgroups Γ such that $M_\Gamma = (H_3 \times \mathbb{R})/\Gamma$ is a compact manifold; any hypersymplectic structure on $\mathfrak{g}_0^h$ induces a hypersymplectic structure on M_Γ, all of them flat. This manifold is a primary Kodaira surface, and it has already been shown in [12] that it carries hypersymplectic structures. On the other hand, the Lie groups G_1^h and G_2^h do not admit discrete cocompact subgroups, since they are not unimodular.

6. Proof of Theorem 23

We will construct explicitly all 4-dimensional Lie algebras carrying a hypersymplectic structure using Theorem 7. In order to do so, we have to determine all the triples $(\mathfrak{g}_+, \nabla, \omega)$, $(\mathfrak{g}_-, \nabla', \omega')$ and the linear isomorphisms $\varphi : \mathfrak{g}_+ \longrightarrow \mathfrak{g}_-$ which satisfy the conditions of this theorem. The Lie algebras $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are 2-dimensional, and therefore they are isomorphic either to $\mathbb{R}^2$ or $\mathfrak{aff}(\mathbb{R})$. The flat torsion-free connections on these Lie algebras which are compatible with the canonical symplectic forms were determined in Section 4. We only have to establish the linear isomorphisms φ which are admissible. We will do this in several steps.

6.1. *Case (A): $\mathfrak{g}_+ = \mathbb{R}^2$ and $\mathfrak{g}_- = \mathbb{R}^2$*

We fix a basis $\{e_1, e_2\}$ of $\mathfrak{g}_+$ and its associated symplectic form $\omega = e^1 \wedge e^2$, where $\{e^1, e^2\}$ is the dual basis. In the same way we fix a basis $\{f_1, f_2\}$ of $\mathfrak{g}_-$ and its associated symplectic form $\omega' = f^1 \wedge f^2$, where $\{f^1, f^2\}$ is the dual basis. In this case there are only two connections to be considered: the connection identically zero and the connection ∇^0 which appears in Proposition 18.

(A1) $\nabla = 0$ and $\nabla' = 0$.

Here $\mathfrak{g} = \mathfrak{g}_+ \bowtie \mathfrak{g}_- = \mathbb{R}^4$ is the abelian 4-dimensional Lie algebra and the hypersymplectic is the canonical one, given as in the statement of the theorem.

(A2) $\nabla = \nabla^0$ and $\nabla' = 0$.

Here $\mathfrak{g} = \mathbb{R}^2 \ltimes \mathbb{R}^2$. In this special case we may simply suppose that the linear isomorphism $\varphi : \mathfrak{aff}(\mathbb{R}) \longrightarrow \mathbb{R}^2$ we are seeking is given by $\varphi(e_i) = f_i, i = 1, 2$. It is easy to see that this isomorphism is compatible with ∇ and ∇' and also with ω and ω'. Hence, we obtain a hypersymplectic structure on $\mathfrak{g}$. Let us identify this Lie algebra. If we denote $e_i := (e_i, 0)$ and $f_i := (0, f_i)$ for $i = 1, 2$, then the only non-zero bracket is $[e_1, f_1] = f_2$. We also have $Je_i = f_i$ and $Ee_i = e_i, Ef_i = -f_i$. By setting

$$v_1 := e_1, \qquad v_2 := f_1, \qquad v_3 := f_2, \qquad v_0 := -e_2,$$

we obtain $[v_1, v_2] = v_3$ and v_3 central; therefore $\mathfrak{g} \cong \mathfrak{g}_0^h$. The complex structure J is given by $Jv_1 = v_2, Jv_3 = v_0$ and the eigenspaces corresponding to E are $\mathfrak{g}_+ = \mathrm{span}\{v_0, v_1\}$, $\mathfrak{g}_- = \mathrm{span}\{v_2, v_3\}$. This complex product structure is equivalent to $\{J^{(0)}, E_\pi^{(0)}\}$.

The hypersymplectic metric on $\mathfrak{g}_0^h$ in this case is homothetic to g_π given by $g_\pi(v_0, v_2) = g_\pi(v_1, v_3) = 1$. Hence, $g_\pi = v^0 \cdot v^2 + v^1 \cdot v^3$. The left-invariant metric g_π on G_0^h is given in terms of the global coordinates by

$$g_\pi = \mathrm{d}t\mathrm{d}y - x\mathrm{d}x\mathrm{d}y + \mathrm{d}x\mathrm{d}z.$$

It is easily seen that the associated torsion-free connection $\nabla^{\mathrm{CP}} = \nabla^{g_\pi}$ is flat and the metric g_π is complete, using Eq. (1).

(A3) $\nabla = 0$ and $\nabla' = \nabla^0$.

Here $\mathfrak{g} = \mathbb{R}^2 \ltimes \mathbb{R}^2$. We may suppose again that the linear isomorphism $\varphi : \mathfrak{aff}(\mathbb{R}) \longrightarrow \mathbb{R}^2$ we are seeking is given by $\varphi(e_i) = f_i$, $i = 1, 2$. It is easy to see that this isomorphism is compatible with ∇ and ∇' and also with ω and ω'. Hence, we obtain a hypersymplectic structure on $\mathfrak{g}$. Let us identify this Lie algebra. If we denote $e_i := (e_i, 0)$ and $f_i := (0, f_i)$ for $i = 1, 2$, then the only non-zero bracket is $[e_1, f_1] = e_2$. We have that $Je_i = f_i$ and $Ee_i = e_i$, $Ef_i = -f_i$. By setting

$$v_1 := e_1, \qquad v_2 := f_1, \qquad v_3 := e_2, \qquad v_0 := f_2,$$

we get $[v_1, v_2] = v_3$ and v_3 central; therefore $\mathfrak{g} \cong \mathfrak{g}_0^h$. The complex structure J is given by $Jv_1 = v_2$, $Jv_3 = v_0$ and the eigenspaces corresponding to E are $\mathfrak{g}_+ = \mathrm{span}\{v_1, v_3\}$ and $\mathfrak{g}_- = \mathrm{span}\{v_0, v_2\}$. This complex product structure is equivalent to $\{J^{(0)}, E_0^{(0)}\}$.

The hypersymplectic metric on $\mathfrak{g}_0^h$ in this case is homothetic to g_0 given by $g_0(v_0, v_1) = 1$, $g_0(v_2, v_3) = -1$. Hence, $g_0 = v^0 \cdot v^1 - v^2 \cdot v^3$. The left-invariant metric g_0 on G_0^h is given in terms of the global coordinates by

$$g_0 = \mathrm{d}t\mathrm{d}x + x\mathrm{d}y^2 - \mathrm{d}y\mathrm{d}z.$$

It is easy to check that the torsion-free connection ∇^{CP} associated with this hypersymplectic structure is flat and that the metric g_0 is complete, using Eq. (1).

(A4) $\nabla = \nabla^0$ and $\nabla' = \nabla^0$.

We are looking for a linear isomorphism $\varphi : \mathfrak{g}_+ \longrightarrow \mathfrak{g}_-$ compatible with ∇ and ∇' and also with ω and ω'. After lengthy computations, we obtain that φ must be of the form $\varphi = \begin{pmatrix} a & 0 \\ b & d \end{pmatrix}$, with $ad = 1$ (in the ordered bases $\{e_1, e_2\}$, $\{f_1, f_2\}$). Hence, we have a hypersymplectic structure on the bicrossproduct Lie algebra $\mathfrak{g} := \mathbb{R}^2 \bowtie \mathbb{R}^2$. Let us denote $e_i := (e_i, 0)$, $f_i := (0, f_i)$ for $i = 1, 2$; the only non-zero bracket is $[e_1, f_1] = -a^2 e_2 + d^2 f_2$. The complex product structure on this Lie algebra is given by

$$Je_1 = af_1 + bf_2, \qquad Je_2 = df_2, \qquad Jf_1 = -de_1 + be_2, \qquad Jf_2 = -ae_2$$

and $Ee_i = e_i$, $Ef_i = -f_i$ for $i = 1, 2$. We will make now a change of basis, setting:

$$v_1 := e_1, \qquad v_2 := af_1 + bf_2, \qquad v_3 := a(-a^2 e_2 + d^2 f_2), \qquad v_0 := -a(de_2 + af_2).$$

Then we have $[v_1, v_2] = v_3$ and hence $\mathfrak{g} \cong \mathfrak{g}_0^h$. The complex structure J is given by $Jv_1 = v_2$, $Jv_3 = v_0$ and the eigenspaces corresponding to E are

$$\mathfrak{g}_+ = \mathrm{span}\left\{v_1, \frac{a^3}{a^6+1} v_3 + \frac{1}{a^6+1} v_0\right\}, \qquad \mathfrak{g}_- = \mathrm{span}\left\{v_2, \frac{1}{a^6+1} v_3 - \frac{a^3}{a^6+1} v_0\right\}.$$

This complex product structure is equivalent to $\{J^{(0)}, E_\theta^{(0)}\}$, where θ is given by $\cos(\theta/2) = \frac{a^3}{\sqrt{a^6+1}}$, $\sin(\theta/2) = \frac{1}{\sqrt{a^6+1}}$. Note that $\theta \neq 0$ and $\theta \neq \pi$. The eigenspaces associated to

$E_\theta^{(0)}$ are $\mathfrak{g}_+ = \text{span}\{U_\theta, v_2\}, \mathfrak{g}_- = \text{span}\{V_\theta, v_3\}$, where $U_\theta = \cos_{\theta/2} v_0 + \sin_{\theta/2} v_1$ and $V_\theta = -\sin_{\theta/2} v_0 + \cos_{\theta/2} v_1$; note that $JU_\theta = V_\theta$ and $Jv_2 = v_3$.

The hypersymplectic metric on $\mathfrak{g}_0^h$ in this case is homothetic to g_θ given by $g_\theta(U_\theta, v_2) = 1, g_0(V_\theta, v_1) = -1$. Hence,

$$g_\theta = (\cos_{\theta/2} v^3 + \sin_{\theta/2} v^0) \cdot v^2 - v^1 \cdot (-\sin_{\theta/2} v^3 + \cos_{\theta/2} v^0)$$

and the left-invariant metric g_0 on G_0^h is given in terms of the global coordinates by

$$\begin{aligned} g_\theta &= -\cos_{\theta/2} x\mathrm{d}y^2 + \cos_{\theta/2} x\mathrm{d}y\mathrm{d}z + \sin_{\theta/2} \mathrm{d}t\mathrm{d}y \\ &\quad - \sin_{\theta/2} x\mathrm{d}x\mathrm{d}y + \sin_{\theta/2} \mathrm{d}x\mathrm{d}z - \cos_{\theta/2} \mathrm{d}t\mathrm{d}x. \end{aligned}$$

It is easy to check that the torsion-free connection ∇^{CP} associated with this hypersymplectic structure is flat and that the metric g_θ is complete, using Eq. (1).

6.2. *Case (B):* $\mathfrak{g}_+ = \mathfrak{aff}(\mathbb{R})$ *and* $\mathfrak{g}_- = \mathbb{R}^2$

We fix a basis $\{e_1, e_2\}$ of $\mathfrak{g}_+$ such that $[e_1, e_2] = e_2$ and its associated symplectic form $\omega = e^1 \wedge e^2$, where $\{e^1, e^2\}$ is the dual basis. In the same way we fix a basis $\{f_1, f_2\}$ of $\mathfrak{g}_-$ and its associated symplectic form $\omega' = f^1 \wedge f^2$, where $\{f^1, f^2\}$ is the dual basis.

(B1) $\nabla = \nabla^1$ and $\nabla' = 0$.

In this case, we have $\mathfrak{g} := \mathfrak{aff}(\mathbb{R}) \ltimes \mathbb{R}^2$. If we denote $e_i := (e_i, 0)$, $f_i := (0, f_i)$ for $i = 1, 2$, then the only non-zero brackets are

$$[e_1, e_2] = e_2, \qquad [e_1, f_1] = -f_1, \qquad [e_1, f_2] = f_2.$$

The complex product structure on this Lie algebra is given by $Je_i = f_i$ and $Ee_i = e_i$, $Ef_i = -f_i$ for $i = 1, 2$. We will make a change of basis, setting $v_0 = -e_1, v_1 = -f_1, v_2 = e_2, v_3 = f_2$. Thus,

$$[v_0, v_1] = v_1, \qquad [v_0, v_2] = -v_2, \qquad [v_0, v_3] = -v_3, \qquad Jv_0 = v_1, \qquad Jv_2 = v_3,$$

and hence $\mathfrak{g} \cong \mathfrak{g}_1^h$. The eigenspaces corresponding to E are the subalgebras $\mathfrak{g}_+ = \text{span}\{v_0, v_2\}$, $\mathfrak{g}_- = \text{span}\{v_1, v_3\}$; this complex product structure is equivalent to $\{J^{(1)}, E_{0,0}^{(1)}\}$.

The hypersymplectic metric on $\mathfrak{g}_1^h$ in this case is homothetic to $g_{0,0}$ given by $g_{0,0}(v_0, v_3) = 1$, $g_{0,0}(v_1, v_2) = -1$; hence, $g_{0,0} = v^0 \cdot v^3 - v^1 \cdot v^2$. The left-invariant metric $g_{0,0}$ on G_1^h is given in terms of the global coordinates by

$$g_{0,0} = \mathrm{e}^t \mathrm{d}t\mathrm{d}z - \mathrm{d}x\mathrm{d}y.$$

It is easy to check that the torsion-free connection ∇^{CP} associated with this hypersymplectic structure is flat. However, the metric $g_{0,0}$ cannot be complete. Indeed, if it were complete, then its restrictions to the eigenspaces of $E_{0,0}^{(1)}$ should be complete. But at least one of these eigenspaces is isomorphic to $\mathfrak{aff}(\mathbb{R})$, and we know from Proposition 21 that any flat torsion-free connection on $\mathfrak{aff}(\mathbb{R})$ compatible with a symplectic form is not complete, a contradiction.

(B2) $\nabla = \nabla^1$ and $\nabla' = \nabla^0$.

Any linear isomorphism $\varphi : \mathfrak{aff}(\mathbb{R}) \longrightarrow \mathbb{R}^2$ compatible with ∇ and ∇' and also with ω and ω' must be of the form $\varphi = \begin{pmatrix} a & 0 \\ b & d \end{pmatrix}$, with $ad = 1$ (in the ordered bases $\{e_1, e_2\}, \{f_1, f_2\}$). With such a φ we have a hypersymplectic structure on the bicrossproduct Lie algebra $\mathfrak{g} := \mathfrak{g}_+ \bowtie \mathfrak{g}_- = \mathfrak{aff}(\mathbb{R}) \bowtie \mathbb{R}^2$. Let us denote $e_i := (e_i, 0)$, $f_i := (0, f_i)$ for $i = 1, 2$; the only non-zero brackets are

$$[e_1, e_2] = e_2, \qquad [e_1, f_1] = -a^2 e_2 - f_1 - 2bdf_2, \qquad [e_1, f_2] = f_2.$$

The complex product structure on this Lie algebra is given by

$$Je_1 = af_1 + bf_2, \qquad Je_2 = df_2, \qquad Jf_1 = -de_1 + be_2, \qquad Jf_2 = -ae_2$$

and $Ee_i = e_i, Ef_i = -f_i$ for $i = 1, 2$. If we set $v_0 := -e_1 + \frac{a^2}{2} f_2, v_1 := -\frac{a^3}{2} e_2 - af_1 - bf_2, v_2 := e_2, v_3 := df_2$ we obtain

$$[v_0, v_1] = v_1, \qquad [v_0, v_2] = -v_2, \qquad [v_0, v_3] = -v_3, \qquad Jv_0 = v_1, \qquad Jv_2 = v_3,$$

and thus $\mathfrak{g} \cong \mathfrak{g}_1^h$. The eigenspaces corresponding to E, in this new basis, are given by

$$\mathfrak{g}_+ = \text{span}\left\{v_0 - \frac{a^3}{2} v_3, v_2\right\}, \qquad \mathfrak{g}_- = \text{span}\left\{v_1 + \frac{a^3}{2} v_2, v_3\right\}.$$

This complex product structure is equivalent to $\{J^{(1)}, E_{0,1}^{(1)}\}$, for all $a \neq 0$. The eigenspaces associated to $E_{0,1}^{(1)}$ are

$$\mathfrak{g}_+ = \text{span}\{v_0 + v_3, v_2\}, \qquad \mathfrak{g}_- = \text{span}\{v_1 - v_2, v_3\}.$$

The hypersymplectic metric on $\mathfrak{g}_1^h$ in this case is homothetic to $g_{0,1}$ given by $g_{0,1}(v_0 + v_3, v_3) = 1$, $g_{0,1}(v_1 - v_2, v_2) = -1$; hence, $g_{0,1} = (v^0 + v^3) \cdot v^3 - (v^1 - v^2) \cdot v^2$. The left-invariant metric $g_{0,1}$ on G_1^h is given in terms of the global coordinates by

$$g_{0,1} = \mathrm{e}^t \mathrm{d}t\mathrm{d}z + \mathrm{e}^{2t}\mathrm{d}z^2 - \mathrm{d}x\mathrm{d}y + \mathrm{e}^{2t}\mathrm{d}y^2.$$

The connection $\nabla^{\text{CP}} = \nabla^{g_{0,1}}$ on $\mathfrak{g}_1^h$ can be easily computed and we can deduce from this computation that its curvature tensor R satisfies

$$R(v_0 + v_3, v_1 - v_2)(v_0 + v_3) = 6v_2, \qquad R(v_0 + v_3, v_1 - v_2)(v_1 - v_2) = 6v_3,$$

and is zero for all the other possibilities. Hence, $g_{0,1}$ is not flat but, as in the case (B1), this metric is not complete.

(B3) $\nabla = \nabla^2$ and $\nabla' = 0$.

In this case, we have $\mathfrak{g} := \mathfrak{aff}(\mathbb{R}) \ltimes \mathbb{R}^2$. If we denote $e_i := (e_i, 0)$, $f_i := (0, f_i)$ for $i = 1, 2$, then the only non-zero brackets are

$$[e_1, e_2] = e_2, \qquad [e_1, f_1] = -\frac{1}{2} f_1, \qquad [e_1, f_2] = \frac{1}{2} f_2, \qquad [e_2, f_1] = -\frac{1}{2} f_2.$$

The complex product structure on this Lie algebra is given by $Je_i = f_i$ and $Ee_i = e_i$, $Ef_i = -f_i$ for $i = 1, 2$. We will make a change of basis, setting $v_0 := 2e_1, v_1 := e_2, v_2 := -2f_1, v_3 := f_2$. Thus,

$$\begin{gathered}[v_0, v_1] = 2v_1, \qquad [v_0, v_2] = -v_2, \qquad [v_0, v_3] = v_3, \qquad [v_1, v_2] = v_3, \\ Jv_0 = -v_2, \qquad Jv_1 = v_3,\end{gathered}$$

and thus $\mathfrak{g} \cong \mathfrak{g}_2^h$. The eigenspaces corresponding to E, in this new basis, are given by $\mathfrak{g}_+ = \text{span}\{v_0, v_1\}$, $\mathfrak{g}_- = \text{span}\{v_2, v_3\}$, and this complex product structure is equivalent to $\{J^{(2)}, E_{0,0}^{(2)}\}$.

The hypersymplectic metric on $\mathfrak{g}_2^h$ in this case is homothetic to $g_{0,0}$ given by $g_{0,0}(v_0, v_3) = g_{0,0}(v_1, v_2) = 1$; hence, $g_{0,0} = v^0 \cdot v^3 + v^1 \cdot v^2$. The left-invariant metric $g_{0,0}$ on G_2^h is given in terms of the global coordinates by

$$g_{0,0} = \mathrm{e}^{-t}\mathrm{d}t\left(\mathrm{d}z - \frac{1}{2}x\mathrm{d}y + \frac{1}{2}y\mathrm{d}x\right) + \mathrm{e}^{-t}\mathrm{d}x\mathrm{d}y.$$

It can be shown that this metric is flat and non-complete.

(B4) $\nabla = \nabla^2$ and $\nabla' = \nabla^0$.

Any linear isomorphism $\varphi : \mathfrak{aff}(\mathbb{R}) \longrightarrow \mathbb{R}^2$ compatible with ∇ and ∇' and also with ω and ω' must be of the form $\varphi = \begin{pmatrix} a & 0 \\ b & d \end{pmatrix}$, with $ad = 1$ (in the ordered bases $\{e_1, e_2\}$, $\{f_1, f_2\}$). Hence, we have a hypersymplectic structure on the bicrossproduct Lie algebra $\mathfrak{g} := \mathfrak{g}_+ \bowtie \mathfrak{g}_- = \mathfrak{aff}(\mathbb{R}) \bowtie \mathbb{R}^2$. Let us denote $e_i := (e_i, 0)$, $f_i := (0, f_i)$ for $i = 1, 2$; the only non-zero brackets are

$$[e_1, e_2] = e_2, [e_1, f_1] = -a^2 e_2 - \frac{1}{2} f_1 - bdf_2, [e_1, f_2] = \frac{1}{2} f_2, [e_2, f_1] = -\frac{1}{2}d^2 f_2.$$

The complex product structure on this Lie algebra is given by

$$Je_1 = af_1 + bf_2, \qquad Je_2 = df_2, \qquad Jf_1 = -de_1 + be_2, \qquad Jf_2 = -ae_2$$

and $Ee_i = e_i, Ef_i = -f_i$ for $i = 1, 2$. We will make now a change of basis, setting $v_0 := 2e_1 - \frac{4}{3}a^2 f_2$, $v_1 := e_2$, $v_2 := -\left(\frac{4}{3}a^3 e_2 + 2af_1 + 2bf_2\right)$, $v_3 := df_2$. Thus,

$$[v_0, v_1] = 2v_1, \qquad [v_0, v_2] = -v_2, \qquad [v_0, v_3] = v_3, \qquad [v_1, v_2] = v_3,$$
$$Jv_0 = -v_2, \qquad Jv_1 = v_3,$$

and thus $\mathfrak{g} \cong \mathfrak{g}_2^h$. The eigenspaces corresponding to E, in this new basis, are given by

$$\mathfrak{g}_+ = \mathrm{span}\left\{v_0 + \frac{4}{3}a^3 v_3, v_1\right\}, \qquad \mathfrak{g}_- = \mathrm{span}\left\{v_2 + \frac{4}{3}a^3 v_1, v_3\right\}.$$

This complex product structure is equivalent to $\{J^{(2)}, E_{0,1}^{(2)}\}$, for all $a \neq 0$, and the eigenspaces associated to $E_{0,1}^{(2)}$ are

$$\mathfrak{g}_+ = \mathrm{span}\{v_0 + v_3, v_1\}, \qquad \mathfrak{g}_- = \mathrm{span}\{v_1 + v_2, v_3\}.$$

The hypersymplectic metric on $\mathfrak{g}_2^h$ in this case is homothetic to $g_{0,1}$ given by $g_{0,1}(v_0 + v_3, v_3) = g_{0,1}(v_1 + v_2, v_1) = 1$; hence, $g_{0,1} = (v^0 + v^3) \cdot v^3 + (v^1 + v^2) \cdot v^1$. The left-invariant metric $g_{0,1}$ on G_1^h is given in terms of the global coordinates by

$$\begin{aligned} g_{0,1} = {} & \mathrm{e}^{-t}\mathrm{d}t\left(\mathrm{d}z - \frac{1}{2}x\mathrm{d}y + \frac{1}{2}y\mathrm{d}x\right) + \mathrm{e}^{-t}\mathrm{d}x\mathrm{d}y + \mathrm{e}^{-4t}\mathrm{d}x^2 \\ & + \mathrm{e}^{-2t}\left(\mathrm{d}z - \frac{1}{2}x\mathrm{d}y + \frac{1}{2}y\mathrm{d}x\right)^2. \end{aligned}$$

The connection $\nabla^{\mathrm{CP}} = \nabla^{g_{0,1}}$ on $\mathfrak{g}_2^h$ can be easily computed and we can deduce from this computation that its curvature tensor R satisfies

$$R(v_0 + v_3, v_1 + v_2)(v_0 + v_3) = -6v_1, \qquad R(v_0 + v_3, v_1 + v_2)(v_1 + v_2) = 6v_3,$$

and is zero for all the other possibilities. Hence, $g_{0,1}$ is not flat but, as in the case (B1), this metric is not complete.

6.3. *Case* (B'): $\mathfrak{g}_+ = \mathbb{R}^2$ *and* $\mathfrak{g}_- = \mathfrak{aff}(\mathbb{R})$

We fix a basis $\{e_1, e_2\}$ of $\mathfrak{g}_+$ and its associated symplectic form $\omega = e^1 \wedge e^2$, where $\{e^1, e^2\}$ is the dual basis. In the same way we fix a basis $\{f_1, f_2\}$ of $\mathfrak{g}_-$ such that $[f_1, f_2] = f_2$ and its associated symplectic form $\omega' = f^1 \wedge f^2$, where $\{f^1, f^2\}$ is the dual basis.

(B1$'$)$\nabla = 0$ and $\nabla' = \nabla^1$.

In this case, we have $\mathfrak{g} := \mathfrak{aff}(\mathbb{R}) \ltimes \mathbb{R}^2$. If we denote $e_i := (e_i, 0)$, $f_i := (0, f_i)$ for $i = 1, 2$, then the only non-zero brackets are

$$[e_1, f_1] = e_1, \qquad [e_2, f_1] = -e_2, \qquad [f_1, f_2] = f_2.$$

The complex product structure on this Lie algebra is given by $Je_i = f_i$ and $Ee_i = e_i$, $Ef_i = -f_i$ for $i = 1, 2$. By setting $v_0 = -f_1$, $v_1 = e_1$, $v_2 = e_2$, $v_3 = f_2$, we get

$$[v_0, v_1] = v_1, \qquad [v_0, v_2] = -v_2, \qquad [v_0, v_3] = -v_3,$$
$$Jv_0 = v_1, \qquad Jv_2 = v_3,$$

and thus $\mathfrak{g} \cong \mathfrak{g}_1^h$. The eigenspaces corresponding to E are the subalgebras $\mathfrak{g}_+ = \operatorname{span}\{v_1, v_2\}$, $\mathfrak{g}_- = \operatorname{span}\{v_0, v_3\}$, and this complex product structure is equivalent to $\{J^{(1)}, E^{(1)}_{\pi,0} = E^{(1)}_{\pi,1}\}$.

The hypersymplectic metric on $\mathfrak{g}_1^h$ in this case is homothetic to $g_{\pi,0}$ given by $g_{\pi,0}(v_1, v_3) = g_{\pi,0}(v_0, v_2) = 1$; hence, $g_{\pi,0} = v^1 \cdot v^3 + v^0 \cdot v^2$. The left-invariant metric $g_{\pi,0}$ on G_1^h is given in terms of the global coordinates by

$$g_{\pi,0} = \mathrm{e}^t \mathrm{d}t\mathrm{d}y + \mathrm{d}x\mathrm{d}z.$$

It is easy to check that the torsion-free connection ∇^{CP} associated with this hypersymplectic structure is flat. However, the metric $g_{\pi,0}$ is not complete.

(B2$'$)$\nabla = \nabla^0$ and $\nabla' = \nabla^1$.

The linear isomorphisms $\varphi : \mathfrak{aff}(\mathbb{R}) \longrightarrow \mathbb{R}^2$ compatible with ∇ and ∇' and also with ω and ω' must be of the form $\varphi = \begin{pmatrix} a & 0 \\ b & d \end{pmatrix}$, with $ad = 1$ (in the ordered bases $\{e_1, e_2\}$, $\{f_1, f_2\}$). For such a φ there is a hypersymplectic structure on the bicrossproduct Lie algebra $\mathfrak{g} := \mathbb{R}^2 \bowtie \mathfrak{aff}(\mathbb{R})$. Let us denote $e_i := (e_i, 0)$, $f_i := (0, f_i)$ for $i = 1, 2$; the only non-zero brackets are

$$[e_1, f_1] = e_1 - 2abe_2 + d^2 f_2, \qquad [e_2, f_1] = -e_2, \qquad [f_1, f_2] = f_2.$$

The complex product structure on this Lie algebra is given by

$$Je_1 = af_1 + bf_2, \qquad Je_2 = df_2, \qquad Jf_1 = -de_1 + be_2, \qquad Jf_2 = -ae_2$$

and $Ee_i = e_i$, $Ef_i = -f_i$ for $i = 1, 2$. We will make now a change of basis, setting $v_0 := \frac{d^2}{2} e_2 - f_1$, $v_1 := de_1 - be_2 + \frac{d^3}{2} f_2$, $v_2 := e_2$, $v_3 := df_2$. Thus,

$$[v_0, v_1] = v_1, \qquad [v_0, v_2] = -v_2, \qquad [v_0, v_3] = -v_3, \qquad Jv_0 = v_1, \qquad Jv_2 = v_3,$$

and thus $\mathfrak{g} \cong \mathfrak{g}_1^h$. The eigenspaces corresponding to E, in this new basis, are

$$\mathfrak{g}_+ = \operatorname{span}\left\{ v_1 - \frac{d^2}{2} v_3, v_2 \right\}, \qquad \mathfrak{g}_- = \operatorname{span}\left\{ -v_0 + \frac{d^2}{2} v_2, v_3 \right\}.$$

This complex product structure is equivalent to $\{J^{(1)}, E_1^{(1)}\}$, for all $d \neq 0$, and the eigenspaces associated to $E_1^{(1)}$ are $\mathfrak{g}_+ = \text{span}\{v_1 + v_3, v_2\}$, $\mathfrak{g}_- = \text{span}\{v_0 + v_2, v_3\}$.

Every hypersymplectic metric on $\mathfrak{g}_1^h$ corresponding to this complex product structure is homothetic to $g_1 = (v^1 + v^3) \cdot v^3 + (v^0 + v^2) \cdot v^2$, which gives rise to a left-invariant metric on G_1^h, given by

$$g_1 = \mathrm{d}x\mathrm{d}z + \mathrm{e}^{2t}\mathrm{d}z^2 + \mathrm{e}^t\mathrm{d}t\mathrm{d}y + \mathrm{e}^{2t}\mathrm{d}y^2.$$

It can be shown that the metric g_1 is flat and not complete.

(B3′)$\nabla = 0$ and $\nabla' = \nabla^2$.

In this case, we have $\mathfrak{g} := \mathfrak{aff}(\mathbb{R}) \ltimes \mathbb{R}^2$. If we denote $e_i := (e_i, 0)$, $f_i := (0, f_i)$ for $i = 1, 2$, then the only non-zero brackets are

$$[e_1, f_1] = \frac{1}{2}e_1, \qquad [e_1, f_2] = \frac{1}{2}e_2, \qquad [e_2, f_1] = -\frac{1}{2}e_2, \qquad [f_1, f_2] = f_2.$$

The complex product structure on this Lie algebra is given by $Je_i = f_i$ and $Ee_i = e_i$, $Ef_i = -f_i$ for $i = 1, 2$. We will make a change of basis, setting $v_0 := 2f_1$, $v_1 := f_2$, $v_2 := 2e_1$, $v_3 := -e_2$. Thus,

$$[v_0, v_1] = 2v_1, \qquad [v_0, v_2] = -v_2, \qquad [v_0, v_3] = v_3, \qquad [v_1, v_2] = v_3,$$
$$Jv_0 = -v_2, \qquad Jv_1 = v_3,$$

and hence $\mathfrak{g} \cong \mathfrak{g}_2^h$. The eigenspaces corresponding to E, in this new basis, are the subalgebras $\mathfrak{g}_+ = \text{span}\{v_2, v_3\}$, $\mathfrak{g}_- = \text{span}\{v_0, v_1\}$. This complex product structure is equivalent to $\{J^{(2)}, E_{\pi,0}^{(2)} = E_{\pi,1}^{(2)}\}$.

The hypersymplectic metric on $\mathfrak{g}_2^h$ in this case is homothetic to $g_{\pi,0}$ given by $g_{\pi,0}(v_1, v_2) = g_{\pi,0}(v_0, v_3) = 1$; hence, $g_{\pi,0} = v^1 \cdot v^2 + v^0 \cdot v^3$. The left-invariant metric $g_{\pi,0}$ on G_1^h is given in terms of the global coordinates by

$$g_{\pi,0} = \mathrm{e}^{-t}\mathrm{d}t\left(\mathrm{d}z - \frac{1}{2}x\mathrm{d}y + \frac{1}{2}y\mathrm{d}x\right) + \mathrm{e}^{-t}\mathrm{d}x\mathrm{d}y.$$

It is easy to show that this metric is flat and non-complete.

(B4′)$\nabla = \nabla^0$ and $\nabla' = \nabla^2$.

Again, any linear isomorphism $\varphi : \mathfrak{aff}(\mathbb{R}) \longrightarrow \mathbb{R}^2$ compatible with ∇ and ∇' and also with ω and ω' must be of the form $\varphi = \begin{pmatrix} a & 0 \\ b & d \end{pmatrix}$, with $ad = 1$ (in the ordered bases $\{e_1, e_2\}$, $\{f_1, f_2\}$). For such a φ there is a hypersymplectic structure on the bicrossproduct Lie algebra $\mathfrak{g} := \mathbb{R}^2 \bowtie \mathfrak{aff}(\mathbb{R})$. Let us denote $e_i := (e_i, 0)$, $f_i := (0, f_i)$ for $i = 1, 2$; the only non-zero brackets are

$$[e_1, f_1] = \frac{1}{2}e_1 - abe_2 + d^2 f_2,\ [e_1, f_2] = \frac{1}{2}a^2 e_2,\ [e_2, f_1] = -\frac{1}{2}e_2,\ [f_1, f_2] = f_2.$$

The complex product structure on this Lie algebra is given by

$$Je_1 = af_1 + bf_2, \qquad Je_2 = df_2, \qquad Jf_1 = -de_1 + be_2, \qquad Jf_2 = -ae_2$$

and $Ee_i = e_i$, $Ef_i = -f_i$ for $i = 1, 2$. We will make now a change of basis, setting $v_0 := -\frac{4}{3}d^2e_2 + 2f_1$, $v_1 := f_2$, $v_2 := 2de_1 - 2be_2 + \frac{4}{3}d^3 f_2$, $v_3 := -ae_2$. Thus,

$$[v_0, v_1] = 2v_1, \qquad [v_0, v_2] = -v_2, \qquad [v_0, v_3] = v_3, \qquad [v_1, v_2] = v_3,$$
$$Jv_0 = -v_2, \qquad Jv_1 = v_3,$$

so that $\mathfrak{g} \cong \mathfrak{g}_2^h$. The eigenspaces corresponding to E, in this new basis, are given by

$$\mathfrak{g}_+ = \operatorname{span}\left\{-v_2 + \frac{4}{3}d^3 v_1, v_3\right\}, \qquad \mathfrak{g}_- = \operatorname{span}\left\{v_0 - \frac{4}{3}d^3 v_3, v_1\right\}.$$

This complex product structure is equivalent to $\{J^{(2)}, E_1^{(2)}\}$, for all $d \neq 0$; the eigenspaces associated to $E_1^{(2)}$ are $\mathfrak{g}_+ = \operatorname{span}\{v_1 + v_2, v_3\}$, $\mathfrak{g}_- = \operatorname{span}\{v_0 + v_3, v_1\}$.

Every hypersymplectic metric on $\mathfrak{g}_2^h$ corresponding to this complex product structure is homothetic to $g_1 = (v^1 + v^2) \cdot v^1 + (v^0 + v^3) \cdot v^3$, which gives rise to a left-invariant metric on G_2^h, given by

$$\begin{aligned} g_1 &= \mathrm{e}^{-t}\mathrm{d}x\mathrm{d}y + \mathrm{e}^{-4t}\mathrm{d}x^2 + \mathrm{e}^{-t}\mathrm{d}t\left(\mathrm{d}z - \frac{1}{2}x\mathrm{d}y + \frac{1}{2}y\mathrm{d}x\right) \\ &\quad + \mathrm{e}^{-2t}\left(\mathrm{d}z - \frac{1}{2}x\mathrm{d}y + \frac{1}{2}y\mathrm{d}x\right)^2. \end{aligned}$$

This metric is flat and not complete.

6.4. *Case (C):* $\mathfrak{g}_+ = \mathfrak{aff}(\mathbb{R})$ *and* $\mathfrak{g}_- = \mathfrak{aff}(\mathbb{R})$

We will use $\mathfrak{g}_+ = \operatorname{span}\{e_1, e_2\}$ with $[e_1, e_2] = e_2$ and $\mathfrak{g}_- = \operatorname{span}\{f_1, f_2\}$ with $[f_1, f_2] = f_2$; the symplectic forms are $\omega = e^1 \wedge e^2$ and $\omega' = f^1 \wedge f^2$. Clearly, none of the connections on $\mathfrak{g}_+$ or $\mathfrak{g}_-$ may be zero, since in that case the Lie algebra would turn out to be abelian.

(C1) $\nabla = \nabla^1$ and $\nabla' = \nabla^1$.

Any linear isomorphism $\varphi : \mathfrak{g}_+ \longrightarrow \mathfrak{g}_-$ compatible with ∇ and ∇' and also with ω and ω'. must be of the form $\varphi = \begin{pmatrix} a & 0 \\ b & d \end{pmatrix}$, with $ad = 1$ (in the ordered bases $\{e_1, e_2\}$, $\{f_1, f_2\}$). For such a φ we obtain a hypersymplectic structure on the bicrossproduct Lie algebra $\mathfrak{g} := \mathfrak{g}_+ \bowtie \mathfrak{g}_- = \mathfrak{aff}(\mathbb{R}) \bowtie \mathfrak{aff}(\mathbb{R})$. Let us denote $e_i := (e_i, 0)$, $f_i := (0, f_i)$ for $i = 1, 2$; the only non-zero brackets are

$$\begin{aligned} &[e_1, e_2] = e_2, \qquad [e_1, f_1] = e_1 - 2abe_2 - f_1 - 2bdf_2, \qquad [e_1, f_2] = f_2, \\ &\quad [e_2, f_1] = -e_2, \qquad [f_1, f_2] = f_2. \end{aligned}$$

The complex product structure on this Lie algebra is given by

$$Je_1 = af_1 + bf_2, \qquad Je_2 = df_2, \qquad Jf_1 = -de_1 + be_2, \qquad Jf_2 = -ae_2$$

and $Ee_i = e_i$, $Ef_i = -f_i$ for $i = 1, 2$. We will make a change of basis, setting

$$\begin{aligned} &v_0 := -\frac{1}{a^2+1}(e_1 + a^2 f_1), \qquad v_1 := \frac{1}{a^2+1}(a(e_1 - f_1) - a^2 be_2 - bf_2), \\ &\quad v_2 := ae_2, \qquad v_3 := f_2. \end{aligned}$$

Thus, we have that

$$\begin{aligned} &[v_0, v_1] = v_1, \qquad [v_0, v_2] = -v_2, \qquad [v_0, v_3] = -v_3, \\ &\quad Jv_0 = v_1, \qquad Jv_2 = v_3, \end{aligned}$$

so that $\mathfrak{g} \cong \mathfrak{g}_1^h$. The eigenspaces corresponding to E, in this new basis, are given by

$$\mathfrak{g}_+ = \operatorname{span}\left\{ v_0 - av_1 - \frac{ab}{a^2+1}v_3, v_2 \right\}, \qquad \mathfrak{g}_- = \operatorname{span}\left\{ v_0 + dv_1 + \frac{b}{a^2+1}v_2, v_3 \right\}.$$

This complex product structure is equivalent to either $\{J^{(1)}, E^{(1)}_{\theta,0}\}$ if $b = 0$ or $\{J^{(1)}, E^{(1)}_{\theta,1}\}$ if $b \neq 0$, in both cases with $\cos\theta = \frac{1-a^2}{1+a^2}$, $\sin\theta = -\frac{2a}{1+a^2}$. Note that $\theta \neq 0$ and $\theta \neq \pi$ (since $a \neq 0$).

On the one hand, the eigenspaces associated to $E^{(1)}_{\theta,0}$ are the subalgebras $\mathfrak{g}_+ = \operatorname{span}\{U_\theta, v_2\}$, $\mathfrak{g}_- = \operatorname{span}\{V_\theta, v_3\}$, where $U_\theta = \cos_{\theta/2} v_0 + \sin_{\theta/2} v_1$ and $V_\theta = -\sin_{\theta/2} v_0 + \cos_{\theta/2} v_1$; note that $JU_\theta = V_\theta$ and $Jv_2 = v_3$. Every hypersymplectic metric on $\mathfrak{g}_1^h$ corresponding to this complex product structure is homothetic to $g_{\theta,0}(U_\theta, v_3) = 1$, $g_{\theta,0}(V_\theta, v_2) = -1$; hence

$$g_{\theta,0} = (\cos_{\theta/2} v^0 + \sin_{\theta/2} v^1)\cdot v^3 - (-\sin_{\theta/2} v^0 + \cos_{\theta/2} v^1)\cdot v^2.$$

From this we obtain that the left-invariant metric $g_{\theta,0}$ on G_1^h is given in terms of the global coordinates by

$$g_{\theta,0} = \cos_{\theta/2} \mathrm{e}^t \mathrm{d}t\mathrm{d}z + \sin_{\theta/2} \mathrm{d}x\mathrm{d}z + \sin_{\theta/2} \mathrm{e}^t \mathrm{d}t\mathrm{d}y - \cos_{\theta/2} \mathrm{d}x\mathrm{d}y.$$

The connection $\nabla^{\mathrm{CP}} = \nabla^{g_{\theta,0}}$ on $\mathfrak{g}_1^h$ can be explicitly computed, and we can deduce from this computation that it is flat. Moreover, this connection is not complete.

On the other hand, the eigenspaces associated to $E^{(1)}_{\theta,1}$ are the subalgebras $\mathfrak{g}_+ = \operatorname{span}\{U_\theta, v_2\}$, $\mathfrak{g}_- = \operatorname{span}\{V_\theta, v_3\}$, where $U_\theta = \cos_{\theta/2} v_0 + \sin_{\theta/2} v_1 + \cos_{\theta/2} v_3$ and $V_\theta = -\sin_{\theta/2} v_0 + \cos_{\theta/2} v_1 - \cos_{\theta/2} v_2$; note that $JU_\theta = V_\theta$ and $Jv_2 = v_3$. Every hypersymplectic metric on $\mathfrak{g}_1^h$ corresponding to this complex product structure is homothetic to $g_{\theta,1}(U_\theta, v_3) = 1$, $g_{\theta,0}(V_\theta, v_2) = -1$; hence

$$\begin{aligned} g_{\theta,1} &= (\cos_{\theta/2} v^0 + \sin_{\theta/2} v^1 + \cos_{\theta/2} v^3)\cdot v^3 \\ &\quad - (-\sin_{\theta/2} v^0 + \cos_{\theta/2} v^1 - \cos_{\theta/2} v^2)\cdot v^2. \end{aligned}$$

The left-invariant metric $g_{\theta,1}$ on G_1^h is given in terms of the global coordinates by

$$\begin{aligned} g_{\theta,1} &= \cos_{\theta/2} \mathrm{e}^t \mathrm{d}t\mathrm{d}z + \sin_{\theta/2} \mathrm{d}x\mathrm{d}z + \cos_{\theta/2} \mathrm{e}^{2t} \mathrm{d}z^2 \\ &\quad + \sin_{\theta/2} \mathrm{e}^t \mathrm{d}t\mathrm{d}y - \cos_{\theta/2} \mathrm{d}x\mathrm{d}y + \cos_{\theta/2} \mathrm{e}^{2t} \mathrm{d}y^2. \end{aligned}$$

The connection $\nabla^{\mathrm{CP}} = \nabla^{g_{\theta,1}}$ on $\mathfrak{g}_1^h$ can be explicitly computed, and we can deduce from this computation that

$$R(U_\theta, V_\theta)U_\theta = 6\cos(\theta/2)v_2, \qquad R(U_\theta, V_\theta)V_\theta = 6\cos(\theta/2)v_3,$$

and is zero for the other possibilities. Therefore, $g_{\theta,1}$ is not flat in this case (since $\theta \neq \pi$). As in previous cases, the metrics $g_{\theta,1}$ are not complete.

(C2) $\nabla = \nabla^1$ and $\nabla' = \nabla^2$ or $\nabla = \nabla^2$ and $\nabla' = \nabla^1$.

In these cases there does not exist any $\varphi : \mathfrak{g}_+ \longrightarrow \mathfrak{g}_-$ compatible with ∇ and ∇'.

(C3) $\nabla = \nabla^2$ and $\nabla' = \nabla^2$.

Any linear isomorphism $\varphi : \mathfrak{g}_+ \longrightarrow \mathfrak{g}_-$ compatible with ∇ and ∇' and also with ω and ω'. must be of the form $\varphi = \begin{pmatrix} a & 0 \\ b & d \end{pmatrix}$, with $ad = 1$ (in the ordered bases $\{e_1, e_2\}$, $\{f_1, f_2\}$). For this φ there is a hypersymplectic structure on the bicrossproduct Lie algebra $\mathfrak{g} := \mathfrak{aff}(\mathbb{R}) \bowtie \mathfrak{aff}(\mathbb{R})$.

Let us denote $e_i := (e_i, 0)$, $f_i := (0, f_i)$ for $i = 1, 2$; the only non-zero brackets are

$$[e_1, e_2] = e_2, \qquad [e_1, f_1] = \frac{1}{2}e_1 - abe_2 - \frac{1}{2}f_1 - bdf_2, \qquad [e_1, f_2] = \frac{1}{2}(a^2e_2 + f_2),$$
$$[e_2, f_1] = -\frac{1}{2}(e_2 + d^2 f_2), \qquad [f_1, f_2] = f_2.$$

The complex product structure on this Lie algebra is given by

$$Je_1 = af_1 + bf_2, \qquad Je_2 = df_2, \qquad Jf_1 = -de_1 + be_2, \qquad Jf_2 = -ae_2$$

and $Ee_i = e_i$, $Ef_i = -f_i$ for $i = 1, 2$. We will make a change of basis, setting

$$v_0 := \frac{2}{a^2+1}\left(e_1 - \frac{2a^3b}{3(a^2+1)}e_2 + a^2 f_1 + \frac{2ab}{3(a^2+1)}f_2\right),$$
$$v_2 := \frac{2}{a^2+1}\left(a(e_1 - f_1) - \frac{a^2b(3a^2+1)}{3(a^2+1)}e_2 - \frac{b(a^2+3)}{3(a^2+1)}f_2\right),$$
$$v_1 := e_2 + f_2, \quad v_3 := -ae_2 + df_2.$$

Thus, we have that

$$[v_0, v_1] = 2v_1, \qquad [v_0, v_2] = -v_2, \qquad [v_0, v_3] = v_3, \qquad [v_1, v_2] = v_3,$$
$$Jv_0 = -v_2, \qquad Jv_1 = v_3,$$

so that $\mathfrak{g} \cong \mathfrak{g}_2^h$. The eigenspaces corresponding to E, in this new basis, are given by

$$\mathfrak{g}_+ = \operatorname{span}\left\{v_0 + av_2 + \frac{2a^2b}{3(a^2+1)}v_3, v_1 - av_3\right\},$$
$$\mathfrak{g}_- = \operatorname{span}\left\{v_0 - dv_2 - \frac{2ab}{3(a^2+1)}v_1, av_1 + v_3\right\}.$$

This complex product structure is equivalent to either $\{J^{(2)}, E^{(2)}_{\theta,0}\}$ if $b = 0$ or $\{J^{(2)}, E^{(2)}_{\theta,1}\}$ if $b \neq 0$, in both cases with $\cos\theta = \frac{1-a^2}{1+a^2}$, $\sin\theta = -\frac{2a}{1+a^2}$. Note that $\theta \neq 0$ and $\theta \neq \pi$ (since $a \neq 0$).

On the one hand, the eigenspaces associated to $E^{(2)}_{\theta,0}$ are the subalgebras $\mathfrak{g}_+ = \operatorname{span}\{U_\theta, \tilde{U}_\theta\}$, $\mathfrak{g}_- = \operatorname{span}\{V_\theta, \tilde{V}_\theta\}$, where $U_\theta = \cos_{\theta/2} v_0 + \sin_{\theta/2} v_2$, $\tilde{U}_\theta = \cos_{\theta/2} v_1 - \sin_{\theta/2} v_3$ and $V_\theta = \sin_{\theta/2} v_0 - \cos_{\theta/2} v_2$, $\tilde{V}_\theta = \sin_{\theta/2} v_1 + \cos_{\theta/2} v_3$. Note that $JU_\theta = V_\theta$ and $J\tilde{U}_\theta = \tilde{V}_\theta$. Every hypersymplectic metric on $\mathfrak{g}_2^h$ corresponding to this complex product structure is homothetic to $g_{\theta,0}(U_\theta, \tilde{V}_\theta) = 1$, $g_{\theta,0}(\tilde{U}_\theta, V_\theta) = -1$; so that

$$g_{\theta,0} = (\cos_{\theta/2} v^0 + \sin_{\theta/2} v^2) \cdot (\sin_{\theta/2} v^1 + \cos_{\theta/2} v^3)$$
$$- (\cos_{\theta/2} v^1 - \sin_{\theta/2} v^3) \cdot (\sin_{\theta/2} v^0 - \cos_{\theta/2} v^2).$$

Hence, the left-invariant metric $g_{\theta,0}$ on G_2^h is given in terms of the global coordinates by

$$g_{\theta,0} = \mathrm{e}^{-t}\mathrm{d}t\left(\mathrm{d}z - \frac{1}{2}x\mathrm{d}y + \frac{1}{2}y\mathrm{d}x\right) + \mathrm{e}^{-t}\mathrm{d}x\mathrm{d}y.$$

Note that $g_{\theta,0}$ does not depend on θ. It can be shown that this metric is flat and not complete.

On the other hand, the eigenspaces associated to $E^{(2)}_{\theta,1}$ are the subalgebras $\mathfrak{g}_+ = \operatorname{span}\{U_\theta, \tilde{U}_\theta\}$, $\mathfrak{g}_- = \operatorname{span}\{V_\theta, \tilde{V}_\theta\}$, where $U_\theta = \cos_{\theta/2} v_0 + \sin_{\theta/2} v_2 + \cos_{\theta/2} v_3$, $\tilde{U}_\theta = \cos_{\theta/2} v_1 - \sin_{\theta/2} v_3$ and $V_\theta = \sin_{\theta/2} v_0 - \cos_{\theta/2} v_2 - \cos_{\theta/2} v_1$, $\tilde{V}_\theta = \sin_{\theta/2} v_1 + \cos_{\theta/2} v_3$. Note that $JU_\theta = V_\theta$ and $J\tilde{U}_\theta = \tilde{V}_\theta$. Every hypersymplectic metric on $\mathfrak{g}_2^h$ corresponding to this complex product structure is homothetic to $g_{\theta,0}(U_\theta, \tilde{V}_\theta) = 1$, $g_{\theta,0}(\tilde{U}_\theta, V_\theta) = -1$; so that

$$\begin{aligned} g_{\theta,1} = {} & (\cos_{\theta/2} v^0 + \sin_{\theta/2} v^2 + \cos_{\theta/2} v^3) \cdot (\sin_{\theta/2} v^1 + \cos_{\theta/2} v^3) \\ & - (\sin_{\theta/2} v^0 - \cos_{\theta/2} v^2 - \cos_{\theta/2} v^1) \cdot (\cos_{\theta/2} v^1 - \sin_{\theta/2} v^3). \end{aligned}$$

The left-invariant metric $g_{\theta,1}$ on G_2^h is given in terms of the global coordinates by

$$\begin{aligned} g_{\theta,1} = {} & \cos_{\theta/2} \mathrm{e}^t \mathrm{d}t\mathrm{d}z + \sin_{\theta/2} \mathrm{d}x\mathrm{d}z + \cos_{\theta/2} \mathrm{e}^{2t} \mathrm{d}z^2 \\ & + \sin_{\theta/2} \mathrm{e}^t \mathrm{d}t\mathrm{d}y - \cos_{\theta/2} \mathrm{d}x\mathrm{d}y + \cos_{\theta/2} \mathrm{e}^{2t} \mathrm{d}y^2. \end{aligned}$$

The connection $\nabla^{\mathrm{CP}} = \nabla^{g_{\theta,1}}$ on $\mathfrak{g}_2^h$ can be explicitly computed, and we can deduce from this computation that

$$R(U_\theta, V_\theta)U_\theta = 6\cos^2(\theta/2)\tilde{U}_\theta, \qquad R(U_\theta, V_\theta)V_\theta = 6\cos^2(\theta/2)\tilde{V}_\theta,$$

and is zero for the other possibilities. Therefore, $g_{\theta,1}$ is not flat in this case (since $\theta \neq \pi$). As in previous cases, the metrics $g_{\theta,1}$ are not complete.

This concludes the proof of the Theorem 23.

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